

Evolutionary Spectral Density Estimate for Symmetric α -Stable Process

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Abstract—In this paper, we study the spectral analysis of a non-stationary Symmetric α -Stable (S. α .S) signals, $1 < \alpha < 2$. This approach consists of constructing an evolutionary spectral measure by analogy to that defined in the spectral theory of second-order processes. The goal of this work is to define the evolutionary spectral density, by considering the class of oscillatory signals (S. α .S). Since, we give an asymptotic unbiased estimator of this density and we show that the variance of this estimator converges to zero.

Keywords—spectral, density, stationary, evolutionary alpha-stable

I. INTRODUCTION

Stable alpha signals have attracted more and more attention in recent decades. In particular, harmonizable processes (SaS) and their properties have been the subject of several studies such as [1–10]. The applications of stable symmetric processes are numerous in diverse fields. We cite here only a few specialties: signal and image processing, communications and radars, economics, electronic applications, hydrology, ... see [11–26].

The definition of the spectral density of alpha stable signals and its estimations in different cases have been studied in many papers [27–31].

In this paper, we consider $(\Omega, \mathcal{F}, P)$ the probability space and $L^0(\Omega)$ the set of real random variables defined on this space. $(E, \mathcal{B}(E), m)$ being the measurable space with measure m , and the subset of E of finite m -measure : $E_0 = \{A \in E : m(A) < \infty\}$.

An independently scattered σ -additive set function

$$M : E_0 \rightarrow L^0(\Omega)$$

such that for each $A \in E_0$, $M(A)$ designates a symmetric α -stable random measure defined on $(E, \mathcal{B}(E))$.

The measure defined on E_0 by $m(A) = [M(A), M(A)]_\alpha$ represents the control measure of M . In this paper we consider that $E = R$ and $\mathcal{B}(E) = \mathcal{B}(R)$ (the borel set).

A random measure M SaS has rotationally invariant (or isotropic) increments if the distribution of $\{e^{i\theta} M(A), A \in E_0\}$, does not depend on θ .

We consider the class of oscillatory process (S. α .S) defined on a family $\mathfrak{R}$ of real functions $\{\varphi_t(\omega)\}$ depending on the instant t , such that the process X admits the following integral representation:

$$X(t) = \int_{-\infty}^{+\infty} \varphi_t(\omega) dM(\omega). \quad (1)$$

where M is a complex-valued random measure and symmetric alpha stable. Their increments are independent and isotropic.

Suppose that, the function $\varphi_t(\omega)$ is defined as an oscillatory function:

$$\varphi_t(\omega) = e^{i\omega t} A_t(\omega),$$

where

$$A_t(\omega) = \int_{-\infty}^{+\infty} e^{it\theta} dK_\omega(\theta). \quad (2)$$

The integral representation considered by Masry and Cambanis in [4], $X(t) = \int_{-\infty}^{+\infty} e^{i\omega t} dM(\omega)$, is a particular case where $A_t(\cdot) = 1$ for all t .

As in [32–35], we define, for the family $\mathfrak{R}$ the function $B_{\mathfrak{R}}(\omega)$ by

$$B_{\mathfrak{R}}(\omega) = \int_{-\infty}^{+\infty} |\theta| |dK_\omega(\theta)|. \quad (3)$$

As [32], we qualify the family $\mathfrak{R}$ of oscillatory functions as semi-stationary if the function $B_{\mathfrak{R}}(\omega)$ is bounded whatever ω , and we define the characteristic width of the family $\mathfrak{R}$ by the following constant

$$B_{\mathfrak{R}} = \left[\sup_{\omega} B_{\mathfrak{R}}(\omega) \right]^{-1} \quad (4)$$

Assuming that $A_t(\cdot) \in L_\alpha(m)$, (i.e. $\int |A_t(x)|^\alpha dm(x) < \infty$) in this case and from the properties of the covariation [1, 4], we have

$$[X(t), X(s)]_\alpha = \int \varphi_t(\omega) (\varphi_s(\omega))^{\alpha-1} dm(\omega). \quad (5)$$

For $s = t$ we may write

$$R(t) = [X(t), X(t)]_\alpha = \int |A_t(\omega)|^\alpha dm(\omega)$$

R cannot be interpreted as the total energy like in the case of the second-order process but by analogy, we can see R as the sum of components $|A_t(\omega)|^\alpha dm(\omega)$ forming the contribution from the frequency ω carried by the process at time t .

As in [37] for the second order process, where he defined the evolutionary spectral measure from the variance, we consider the measure

$$dH_t(\omega) = |A_t(\omega)|^\alpha dm(\omega) \quad (6)$$

as the evolutionary spectral measure at time t according to the family $\mathfrak{R} = \{\varphi_t(\omega)\}_t = \{e^{i\omega t} A_t(\omega)\}$.

As in the paper [33], we can give an interpretation of oscillatory processes under the aspect of time-dependent filters when we suppose that $A_t(\omega)$ is Fourier transform of a filter $h_t : A_t(\omega) = \int_{-\infty}^{\infty} e^{i\omega u} g_t(u) du$. From (1), we obtain $X(t) = \int_{-\infty}^{\infty} L_{t-u}(\omega) g_t(u) du$ with $L_t = \int_{-\infty}^{\infty} e^{i\omega} dM(\omega)$.

Thus, $X(t)$ can be interpreted as the transformation of a stationary alpha stable signal by a time-varying filter g_t . Thus, the evolutionary spectrum at time t , $|A_t(\omega)|^\alpha dm(\omega)$, can be seen as the spectrum of a stationary alpha stable signal (in the sense defined in [4]) where the filter g_t is fixed representing its state at time t .

It is now necessary to “standardize” the functions $A_t(\omega)$ so that, for all ω , we suppose that $A_0(\omega) = 1$. This condition allows $dm(\omega)$ represent the evolutionary spectrum at time $t = 0$. Assume that $\int_{-\infty}^{\infty} |dK_\omega(\theta)| = 1$ and suppose that measure $m(\omega)$ is absolutely continuous with respect to Lebesgue measure. We obtain

$$dH_t(\omega) = h_t(\omega) d\omega \quad (7)$$

$h_t(\omega)$ is called the “evolutionary spectral density”.

The organization of this paper is subject to the following plan: the section II considers the definition of a periodogram by observing the process X on the interval $[0, T]$. Afterwards we introduce the modified periodogram, $\hat{I}_{t,T}(\omega) = C_{\alpha,p} |I_{t,T}(\omega)|^p$, with $0 < p < \frac{\alpha}{2}$. We prove that the modified periodogram is an estimator of $\{h_t(\omega)\}_\alpha^{\frac{p}{\alpha}}$ and it is asymptotically unbiased but it is not consistent. In the section III, we give an estimator by smoothing the modified periodogram using a spectral window $W_T(t)$. We show that the smoothed estimate $\hat{f}_{t,T}(\omega)$ is asymptotically unbiased and consistent of $\{h_t(\omega)\}_\alpha^{\frac{p}{\alpha}}$. Finally, we show the convergence in mean quadratic of the $\hat{f}_{t,T}(\omega)$ to $\{h_t(\omega)\}_\alpha^{\frac{p}{\alpha}}$ and the convergence in probability of the $(\hat{f}_{t,T}(\omega))^{\frac{\alpha}{p}}$ to $h_t(\omega)$.

II. THE PERIODOGRAM

Given a sample record of $\{X(t)\}$, for $0 \leq t \leq T$. We consider the following periodogram:

$$I_{t,T}(\omega) =$$

$$Re \int_{t-T}^{t+T} g_{\alpha,T}(u) X(t-u) \exp(-i\omega\{t-u\}) du, \quad (8)$$

where $\{g_{\alpha,T}(u)\}$ is a filter whose the transfer function is defined by

$$\gamma_{t,T}(\omega) = \int_{t-T}^{t+T} g_{\alpha,T}(u) e^{-i\omega u} du, \quad (9)$$

and for a family $\mathfrak{R}$ of functions $\{A_t(\omega) e^{i\omega t}\}$ we define by

$$\Gamma_{t,\omega,T}(\theta) = \int_{t-T}^{t+T} g_{\alpha,T}(u) \left\{ \frac{A_{t-u}(\omega)}{A_t(\omega)} \right\} e^{-iu\theta} du \quad (10)$$

the generalized transfer function of the filter $\{g_{\alpha,T}(u)\}$. $\{g_{\alpha,T}(u)\}$ is chosen such that:

$$\int_{t-T}^{t+T} |u| |g_{\alpha,T}(u)| du = B_{g_{\alpha,T}}$$

$$\lim_{T \rightarrow \infty} \int_{t-T}^{t+T} |u| |g_{\alpha,T}(u)| du = B_{g_\alpha} < +\infty \quad (11)$$

$$\lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} |\gamma_{t,T}(\omega)|^\alpha d\omega = 1$$

$$\text{and } \lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} |\gamma_{t,T}(\omega)|^{\frac{\alpha}{2}} d\omega = 0. \quad (12)$$

We state an important theorem proved in [1] that we use in the sequel.

Theorem 1.

Let M be a complex SαS measure with independent and isotropic increments, and control measure m then

$$E \exp\{i \int f dM\} = \exp\{-C_\alpha \int |f|^\alpha dm\}$$

for $f \in L_\alpha(m)$ (i. e. $\int |f(x)|^\alpha dm(x) < \infty$), where $C_\alpha = (2\pi)^{-1} \int_{-\pi}^{\pi} |\cos \theta|^\alpha d\theta$.

In virtue of the theorem 1 we have

$$E[\exp\{ir I_{t,T}(\omega)\}] = \exp(-C_\alpha |r|^\alpha \psi_{t,T}(\omega)) \quad (13)$$

where $C_\alpha = (\alpha\pi)^{-1} \int_0^\pi |\cos \theta|^\alpha d\theta$ and $\psi_{t,T}(\omega)$ is given by

$$\psi_{t,T}(\omega) = \int_{-\infty}^{+\infty} \left| \int_{t-T}^{t+T} g_{\alpha,T}(u) \varphi_{t-u}(v) \exp(-i\omega\{t-u\}) du \right|^\alpha dm(v). \quad (14)$$

Now we modify $I_{t,T}(\omega)$ as fractional power periodogram multiplying by a normalization constant:

$$\hat{I}_{t,T}(\omega) = C_{\alpha,p} |I_{t,T}(\omega)|^p \quad (15)$$

where $0 < p < \frac{\alpha}{2}$ and the normalization constant $C_{\alpha,p}$ is given by

$$C_{\alpha,p} = B_p [F_{p,\alpha} [C_\alpha]^{p/\alpha}]^{-1}, \quad (16)$$

where

$$B_p = \int_{-\infty}^{+\infty} \frac{1 - \cos(u)}{|u|^{1+p}} du, \text{ and } F_{p,\alpha} = \int_{-\infty}^{+\infty} \frac{1 - e^{-|u|^\alpha}}{|u|^{1+p}} du.$$

Theorem 2.

For all real number ω , we have

$$E[\hat{I}_{t,T}(\omega)] = [\psi_{t,T}(\omega)]^{\frac{p}{\alpha}}$$

where $\psi_{t,T}(\omega)$ is defined in (14).

Proof:

Using the following identity

$$|x|^p = B_p^{-1} \int_{-\infty}^{+\infty} \frac{1 - \cos(xu)}{|u|^{1+p}} du = B_p^{-1} \operatorname{Re} \int_{-\infty}^{+\infty} \frac{1 - e^{-iux}}{|u|^{1+p}} du. \quad (17)$$

and substituting x by $I_{t,T}(\omega)$, we obtain

$$\hat{I}_{t,T}(\omega) = \frac{1}{F_{p,\alpha}[C_\alpha]^{p/\alpha}} \operatorname{Re} \int_{-\infty}^{+\infty} \frac{1 - e^{-i u I_{t,T}(\omega)}}{|u|^{1+p}} du.$$

Thus, from (8) we have

$$E[\hat{I}_{t,T}(\omega)] = \frac{1}{F_{p,\alpha}[C_\alpha]^{p/\alpha}} \int_{-\infty}^{+\infty} \frac{1 - e^{-|u|^\alpha C_\alpha \psi_{t,T}(\omega)}}{|u|^{1+p}} du. \quad (18)$$

Let $v = (C_\alpha \psi_{t,T}(\omega))^{\frac{1}{\alpha}} u$ and using (12) we get

$$E[\hat{I}_{t,T}(\omega)] = (\psi_{t,T}(\omega))^{\frac{p}{\alpha}}. \quad (19)$$

Using the definition in [33–37], $u(x)$ is a “pseudo δ –function of order L with respect to $v(x)$ ”, if for any k , we have

$$|\int_{-\infty}^{+\infty} u(x)v(x+k)dx - v(k) \int_{-\infty}^{+\infty} u(x)dx| < L. \quad (20)$$

Theorem 3.

Consider a semi stationary family $\{\mathfrak{R}\}$ and its characteristic width $B_{\mathfrak{R}}$. Then, for all t and ω the function $\{e^{it\phi} dK_\omega(\theta)\}$ is a “pseudo δ –function of order $(B_{g_{\alpha,T}}/B_{\mathfrak{R}})$ with respect to the transfer function $\gamma_{t,T}(\theta)$.”

Proof:

For any k , we can write $\gamma_{t,T}(\theta + k) = \gamma_{t,T}(k) + \theta \gamma'_{t,T}(k + \eta(\theta)\theta)$, where, for each θ , $0 < \eta(\theta) < 1$, then

$$\int_{-\infty}^{+\infty} e^{it\theta} \gamma_{t,T}(\theta + k) dK_\omega(\theta) = \gamma_{t,T}(k) \int_{-\infty}^{+\infty} e^{it\theta} dK_\omega(\theta) + \int_{-\infty}^{+\infty} \theta e^{it\theta} \gamma'_{t,T}(k + \eta(\theta)\theta) dK_\omega(\theta).$$

The last integral of above equality was bounded as follows

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} \theta e^{it\theta} \gamma'_{t,T}(k + \eta(\theta)\theta) dK_\omega(\theta) \right| \\ & \leq \sup_{\theta} |\gamma'_{t,T}(\theta)| \int_{-\infty}^{+\infty} |\theta| dK_\omega(\theta) \\ & \leq \sup_{\theta} |\gamma'_{t,T}(\theta)| \frac{1}{B_{\mathfrak{R}}}. \end{aligned}$$

Since $|\gamma'_{t,T}(\omega)| \leq \int_{t-T}^{t+T} |u| |g_{\alpha,T}(u)| du = B_{g_{\alpha,T}}$, the result of this theorem follows.

Theorem 4.

Let $\{g_{\alpha,T}(u)\}$ be a filter satisfying (10) and $\Gamma_{t,\omega,T}(\theta)$ its “generalized transfer function with respect to a semi-stationary family $\mathfrak{R}$ of characteristic width $B_{\mathfrak{R}}$.” We choose $\{g_{\alpha,T}(u)\}$ so that $B_{g_{\alpha,T}} < \varepsilon B_{\mathfrak{R}}$ for all T , then

$$|A_t(\omega)| |\Gamma_{t,\omega,T}(\theta) - \gamma_{t,T}(\theta)| < \frac{B_{g_{\alpha,T}}}{B_{\mathfrak{R}}} \leq \varepsilon \text{ for all } t, \omega, \theta.$$

Proof:

We have, from (11)

$$A_t(\omega) \Gamma_{t,\omega,T}(\theta) = \int_{t-T}^{t+T} g_{\alpha,T}(u) A_{t-u}(\omega) e^{-iu\theta} du \quad (21)$$

Using the expression of $A_{t-u}(\omega)$, the transfer function of $\{g_{\alpha,T}(u)\}$ and interchanging the order of integration, we obtain

$$A_t(\omega) \Gamma_{t,\omega,T}(\theta) = \int_{-\infty}^{+\infty} e^{it\phi} \gamma_{t,T}(\theta + \phi) dK_\omega(\phi). \quad (22)$$

However, according to above Theorem, $\{e^{it\phi} dK_\omega(\phi)\}$ is “a pseudo δ –function of order $(\frac{B_{g_{\alpha,T}}}{B_{\mathfrak{R}}})$ with respect to $\gamma_{t,T}(\phi)$.” Therefore

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} e^{it\phi} \gamma_{t,T}(\phi + \theta) dK_\omega(\phi) \right. \\ & \quad \left. - \gamma_{t,T}(\theta) \int_{-\infty}^{+\infty} e^{it\phi} dK_\omega(\phi) \right| < \frac{B_{g_{\alpha,T}}}{B_{\mathfrak{R}}} \\ & \leq \varepsilon. \end{aligned}$$

The result follows from (2).

The constant B_X , defined by $B_X = \sup_{\mathfrak{R} \in \ell} \{B_{\mathfrak{R}}\}$ is called “the characteristic width of the process $\{X(t)\}$.” Here, ℓ define the set of semi-stationary families. Using (9), we obtain

$$\psi_{t,T}(\omega) = \int_{-\infty}^{+\infty} \left| \int_{t-T}^{t+T} g_{\alpha,T}(u) A_{t-u}(v) \exp(i\{v - \omega\}\{t - u\}) du \right|^\alpha dm(v),$$

Denoting by $r(t, v, \omega, T) = \Gamma_{t,v+\omega,T}(v) - \gamma_{t,T}(v)$. From the theorem 4 we have

$$|r(t, v, \omega, T)| < \left(\frac{B_{g_{\alpha,T}}}{B_X}\right) \frac{1}{|A_t(v+\omega)|} < \varepsilon / |A_t(v+\omega)|. \quad (23)$$

On the other hand, from (9) and (11) we get

$$\begin{aligned} & |\psi_{t,T}(\omega) - \int_{-\infty}^{+\infty} |A_t(v+\omega)|^\alpha |\gamma_{t,T}(v)|^\alpha dm(v+\omega)| \\ & = \left| \int_{-\infty}^{+\infty} [|\Gamma_{t,v+\omega,T}(v)|^\alpha - |\gamma_{t,T}(v)|^\alpha] \times \right. \\ & \quad \left. |A_t(v+\omega)|^\alpha dm(v+\omega) \right|. \end{aligned} \quad (24)$$

Substituting $\Gamma_{t,v+\omega,T}(v)$ by its amount and using the following equality proved in [4]

$$||x+y|^p - |x|^p - |y|^p| \leq 2|xy|^{\frac{p}{2}}, \text{ for all real } x, y \text{ and } 0 < p \leq 2,$$

which implies $||x+y|^r - |x|^r| \leq 2|xy|^{r/2} + |y|^r$, we get

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} [|\Gamma_{t,v+\omega,T}(v)|^\alpha - |\gamma_{t,T}(v)|^\alpha] |A_t(v+\omega)|^\alpha dm(v+\omega) \right| \\ & \leq 2 \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^{\alpha/2} |r(t, v, \omega, T)|^{\alpha/2} \times |A_t(v+\omega)|^\alpha dm(v+\omega) \\ & \quad + \int_{-\infty}^{+\infty} |r(t, v, \omega, T)|^\alpha |A_t(v+\omega)|^\alpha dm(v+\omega) \end{aligned}$$

We majorize the second integral denoted I_2 by:

$$|I_2| \leq \int_{-\infty}^{+\infty} \left| \frac{\varepsilon}{A_t(v+\omega)} \right|^\alpha |A_t(v+\omega)|^\alpha dm(v+\omega) \\ \leq \int_{-\infty}^{+\infty} |\varepsilon|^\alpha dm(v+\omega)$$

because $r(t, v, \omega, T) < \varepsilon/|A_t(v+\omega)|$ and since $dH_t(\omega) = |A_t(\omega)|^\alpha dm(\omega)$ then $dH_0(\omega) = |A_0(\omega)|^\alpha dm(\omega) = dm(\omega)$, (i.e. $dm(v)$ is the evolutionary spectrum at zero time), hence

$$I_2 = O(\varepsilon^\alpha).$$

For the first integral, let the set Ω be defined by $\Omega = \{v; |\gamma_{t,T}(v)|^{\alpha/2} |A_t(v+\omega)|^{\alpha/2} \leq 1\}$ which implies

$$|I_1| \leq |\varepsilon|^{\alpha/2} \left\{ \int_{\Omega} |\gamma_{t,T}(v)|^{\alpha/2} |A_t(v+\omega)|^{\alpha/2} dm(v+\omega) \right. \\ \left. + \int_{\Omega^c} |\gamma_{t,T}(v)|^{\alpha/2} |A_t(v+\omega)|^{\alpha/2} dm(v+\omega) \right\}$$

then, from Cauchy-Schwartz inequality applied to the second integral of the last inequality we have

$$|I_1| \leq |\varepsilon|^{\alpha/2} \left\{ \int_{\Omega} dm(v+\omega) + \int_{\Omega^c} |\gamma_{t,T}(v)|^\alpha dm(v+\omega) \int_{\Omega^c} |A_t(v+\omega)|^\alpha dm(v+\omega) \right\}.$$

The first integral is bounded. For the second integral i.e. $\int_{\Omega^c} |\gamma_{t,T}(v)|^{\frac{\alpha}{2}} |A_t(v+\omega)|^{\frac{\alpha}{2}} dm(v+\omega)$ is finite because $|A_t(v+\omega)| \leq \int_{-\infty}^{+\infty} |dK_\omega(\theta)| = 1$ and

$$\int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha dm(v+\omega) \\ = \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha h_0(v+\omega) dv \\ \leq \sup h_0(v) \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha dv$$

from (13) we conclude that $I_1 + I_2 = O(\varepsilon^{\alpha/2})$. So we have the following result.

Theorem 5.

$$\psi_{t,T}(\omega) = \int_{-\infty}^{+\infty} |A_t(v+\omega)|^\alpha |\gamma_{t,T}(v)|^\alpha dm(v+\omega) \\ + O(\varepsilon^{\alpha/2}) \\ = \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha dH_t(v+\omega) + O(\varepsilon^{\alpha/2})$$

with $B_{g_{\alpha,T}}$ sufficiently small relative to B_X .

Proof:

From (9), re-writing the above result in term of $h_t(v)$

$$\psi_{t,T}(\omega) = \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha h_t(v+\omega) dv + O(\varepsilon^{\alpha/2}). \quad (25)$$

Now suppose that $\{g_{\alpha,T}(u)\}$ is chosen so that $B_{g_{\alpha,T}} \leq B_X \varepsilon$ and $r(t, v, \omega, T) < \left(\frac{B_{g_{\alpha,T}}}{B_X} \right) \frac{1}{|A_t(v+\omega)|}$.

As in [33–37], for give an estimate of $h_t(v)$ we suppose that, $|\gamma_{t,T}(v)|^\alpha$ is a “pseudo δ – function with respect to $h_t(v)$ (the order being $\left(\frac{B_{g_{\alpha,T}}}{B_X} \right)$ ” i.e.

$$\left| \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha h_t(v+\omega) dv - h_t(\omega) \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha dv \right| < \frac{B_{g_{\alpha,T}}}{B_X} \\ \leq \varepsilon.$$

Thus from (20) we have

$$|\psi_{t,T}(\omega) - \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha h_t(v+\omega) dv| \leq k(\varepsilon)^{\alpha/2}$$

$$\left| \psi_{t,T}(\omega) - h_t(\omega) \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha dv \right| \leq k(\varepsilon)^{\alpha/2} + \varepsilon \\ \leq k'(\varepsilon)^{\alpha/2}$$

where k is a constant $k' = k + \varepsilon^{\alpha/2}$, so

$$\psi_{t,T}(\omega) = h_t(\omega) \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^\alpha dv + O(\varepsilon^{\alpha/2})$$

$$\lim_{T \rightarrow \infty} \psi_{t,T}(\omega) = h_t(\omega) + O(\varepsilon^{\alpha/2}).$$

Theorem 7.

Let ω belongs to $]-\pi, +\pi[$ then we have

$$\lim_{T \rightarrow \infty} E[\hat{I}_{t,T}(\omega)] = (h_t(\omega))^{\frac{p}{\alpha}} + O(\varepsilon^{p/2})$$

Proof:

Using the proposition 2 we get

$$E[\hat{I}_{t,T}(\omega)] - (h_t(\omega))^{\frac{p}{\alpha}} = (\psi_{t,T}(\omega))^{\frac{p}{\alpha}} - (h_t(\omega))^{\frac{p}{\alpha}}.$$

Using the inequality: $|x^i - y^i| \leq |x - y|^i$, $x, y > 0$ and $0 < i \leq 1$ we deduce that

$$\left| (\psi_{t,T}(\omega))^{\frac{p}{\alpha}} - (h_t(\omega))^{\frac{p}{\alpha}} \right| \leq |\psi_{t,T}(\omega) - h_t(\omega)|^{\frac{p}{\alpha}} \quad (26)$$

since $\lim_{T \rightarrow \infty} \psi_{t,T}(\omega) = h_t(\omega) + O(\varepsilon^{\alpha/2})$, the result follows.

Theorem 6.

Let ω belongs to $]-\pi, \pi[$ and $0 < p < \frac{\alpha}{2}$, then we have

$$\text{var}[\hat{I}_{t,T}(\omega)] = V_{\alpha,p}(\psi_{t,T}(\omega))^{\frac{2p}{\alpha}}$$

$$\lim_{T \rightarrow \infty} \text{var}[\hat{I}_{t,T}(\omega)] = V_{\alpha,p}(h_t(\omega))^{\frac{2p}{\alpha}} + O(\varepsilon^p)$$

where $V_{\alpha,p} = \frac{C_{\alpha,p}^2}{C_{\alpha,2p}} - 1$.

Proof:

Using the equality (13), putting $v = [C_\alpha \psi_{t,T}(\omega)]^{\frac{1}{\alpha}} u$, and similarly to proof of (14) we get

$$E[\hat{I}_{t,T}(\omega)]^2 = C_{\alpha,p}^2 C_{2p,\alpha}^{-1} [\psi_{t,T}(\omega)]^{\frac{2p}{\alpha}}$$

thus we obtain

$$\text{var}[\hat{I}_{t,T}(\omega)] = V_{\alpha,p}[\psi_{t,T}(\omega)]^{\frac{2p}{\alpha}}$$

using the inequality (21) with $\frac{2p}{\alpha}$ rather than $\frac{p}{\alpha}$, we get

$$\begin{aligned} & \left| V_{\alpha,p}[\psi_{t,T}(\omega)]^{\frac{2p}{\alpha}} - V_{\alpha,p}[h_t(\omega)]^{\frac{2p}{\alpha}} \right| \\ & \leq |V_{\alpha,p}| |\psi_{t,T}(\omega) - h_t(\omega)|^{\frac{2p}{\alpha}} \end{aligned}$$

and since $\lim_{T \rightarrow \infty} \psi_{t,T}(\omega) = h_t(\omega) + O(\varepsilon^{\alpha/2})$, we conclude that

$$\lim_{T \rightarrow \infty} \text{var}[\hat{f}_{t,T}(\omega)] = V_{\alpha,p}[h_t(\omega)]^{\frac{2p}{\alpha}} + O(\varepsilon^p).$$

III. SMOOTHING THE MODIFIED PERIODOGRAM

The variance of the modified periodogram is asymptotically proportional to, $[h_t(\omega)]^{\frac{2p}{\alpha}}$, the spectral density to the power $\frac{2p}{\alpha}$. So it does not converge to zero (non-consistent estimator). To overcome this difficulty we proceed as is customary in time series to smooth the last estimator by a spectral window $W_T(t)$ depending on the parameter M_T :

$$\hat{f}_{t,T}(\omega) = \int_{-\infty}^{+\infty} W_T(\omega - u) \hat{f}_{t,T}(u) du \quad (27)$$

where

$$W_T(t) = M_T W(M_T t), \quad (28)$$

The smoothing parameter M_T satisfies $\lim_{T \rightarrow +\infty} M_T = +\infty$ and $\lim_{T \rightarrow +\infty} \frac{M_T}{T} = 0$, and W is a positive function satisfying $W(-t) = W(t)$, $W(t) = 0$ for $|t| < 1$ and

$$\int_{-1}^{+1} W(t) dt = 1.$$

Smoothing the periodogram reduces the variance of the estimator and gives a good estimate of the spectrum. Indeed, smoothing by a spectral window with well-chosen parameter softens the variation of the spectrum relative to the spectral peaks present in the periodogram.

Theorem 8.

Let ω real number different to zero. Then

$$E[\hat{f}_{t,T}(\omega)] \simeq [h_t(\omega)]^{p/\alpha} + O\left(\frac{p}{\varepsilon^2}\right).$$

$\hat{f}_{t,T}(\omega)$ is an asymptotically unbiased estimate of $[h_t(\omega)]^{p/\alpha}$.

Proof:

Putting $x = M_T(\omega - u)$ in (22), we obtain

$$E[\hat{f}_{t,T}(\omega)] = \int_{-1}^{+1} W(x) E\left[\hat{f}_{t,T}\left(\omega - \frac{x}{M_T}\right)\right] dx.$$

From (12) we have

$$E[\hat{f}_{t,T}(\omega)] = \int_{-1}^{+1} W(x) \left[\psi_{t,T}\left(\omega - \frac{x}{M_T}\right) \right]^{\frac{p}{\alpha}} dx,$$

where $\psi_{t,T}$ is defined in (7). Using (2) and (9) and putting $v - \omega = v'$

$$\psi_{t,T}\left(\omega - \frac{x}{M_T}\right) = \int_{-\infty}^{+\infty} \left| \Gamma_{t,v+\omega,T}\left(v + \frac{x}{M_T}\right) A_t\left(v + \frac{x}{M_T}\right) \right|^{\alpha} dm(v + \omega)$$

so

$$\begin{aligned} & \left| \psi_{t,T}\left(\omega - \frac{x}{M_T}\right) - \int_{-\infty}^{+\infty} |A_t(v + \omega)|^{\alpha} \left| \gamma_{t,T}\left(v + \frac{x}{M_T}\right) \right|^{\alpha} dm(v + \omega) \right| \\ & = \left| \int_{-\infty}^{+\infty} \left(\left| \Gamma_{t,v+\omega,T}\left(v + \frac{x}{M_T}\right) \right|^{\alpha} - \left| \gamma_{t,T}\left(v + \frac{x}{M_T}\right) \right|^{\alpha} \right) |A_t(v + \omega)|^{\alpha} dm(v + \omega) \right| \\ & \leq 2 \int_{-\infty}^{+\infty} \left| \gamma_{t,T}\left(v + \frac{x}{M_T}\right) \right|^{\alpha/2} |r(t, v, \omega, x, T)|^{\alpha/2} |A_t(v + \omega)|^{\alpha} dm(v + \omega) \\ & + \int_{-\infty}^{+\infty} |r(t, v, \omega, x, T)|^{\alpha} |A_t(v + \omega)|^{\alpha} dm(v + \omega) \\ & \triangleq I'_1 + I'_2 \end{aligned}$$

where

$$r(t, v, \omega, x, T) = \Gamma_{t,v+\omega,T}\left(v + \frac{x}{M_T}\right) - \gamma_{t,T}\left(v + \frac{x}{M_T}\right) \quad (29)$$

We know that

$$\begin{aligned} & A_t(v + \omega) \Gamma_{t,v+\omega,T}\left(v + \frac{x}{M_T}\right) = \\ & \int_{t-T}^{t+T} g_{\alpha,T}(u) A_{t-u}(v + \omega) e^{-iu(v + \frac{x}{M_T})} du = \\ & \int_{t-T}^{t+T} \int_{-\infty}^{+\infty} g_{\alpha,T}(u) e^{i(t-u)\phi} e^{-i(v + \frac{x}{M_T})\phi} dK_{v+\omega}(\phi) du \\ & = \int_{-\infty}^{+\infty} \left(\int_{t-T}^{t+T} g_{\alpha,T}(u) e^{-iu(\phi + v + \frac{x}{M_T})} du \right) e^{it\phi} dK_{v+\omega}(\phi) \\ & = \int_{-\infty}^{+\infty} \gamma_{t,T}\left(\phi + v + \frac{x}{M_T}\right) e^{it\phi} dK_{v+\omega}(\phi), \end{aligned}$$

using the inequality (15) and the lemma 3 we have

$$\left| \int_{-\infty}^{+\infty} \gamma_{t,T}\left(\phi + v + \frac{x}{M_T}\right) e^{it\phi} dK_{v+\omega}(\phi) - \gamma_{t,T}\left(v + \frac{x}{M_T}\right) \int_{-\infty}^{+\infty} e^{it\phi} dK_{v+\omega}(\phi) \right| < \varepsilon$$

then

$$\left| A_t(v + \omega) \Gamma_{t,v+\omega,T}\left(v + \frac{x}{M_T}\right) - \gamma_{t,T}\left(v + \frac{x}{M_T}\right) A_t(v + \omega) \right| < \varepsilon$$

$$\left| \Gamma_{t,v+\omega,T}\left(v + \frac{x}{M_T}\right) - \gamma_{t,T}\left(v + \frac{x}{M_T}\right) \right| < \frac{\varepsilon}{|A_t(v + \omega)|}$$

so

$$|r(t, v, \omega, x, T)| < \varepsilon / |A_t(v + \omega)| \quad (30)$$

hence

$$|I'_2| \leq \int_{-\infty}^{+\infty} |\varepsilon|^{\alpha} dm(v + \omega)$$

then

$$I'_2 = O(\varepsilon^{\alpha}). \quad (31)$$

For the first integral I'_1 , let the set Ω_T be defined by

$$\Omega_T = \left\{ v \in R; \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha/2} |A_t(v + \omega)|^{\alpha/2} \leq 1 \right\}$$

from (25) we obtain

$$|I'_1| \leq |\varepsilon|^{\alpha/2} \left\{ \int_{\Omega_T} dm(v + \omega) + \int_{\Omega_T^c} \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha/2} |A_t(v + \omega)|^{\alpha/2} dm(v + \omega) \right\} \quad (32)$$

then, from the Cauchy-Schwartz's inequality the second integral is bounded as follows:

$$|I'_1| \leq |\varepsilon|^{\alpha/2} \left\{ \int_{-\infty}^{+\infty} dm(v + \omega) + \int_{\Omega_T^c} \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha} dm(v + \omega) \int_{\Omega_T^c} |A_t(v + \omega)|^{\alpha} dm(v + \omega) \right\}.$$

The first integral in right side of the above inequality is finite since $dm(v)$ is the evolutionary spectrum at zero time. The second integral is finite because $|A_t(v + \omega)| \leq \int_{-\infty}^{+\infty} |dK_{\omega+v}(\theta)| = 1$ and

$$\begin{aligned} & \int_{\Omega_T^c} \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha} dm(v + \omega) \\ & \leq \int_{-\infty}^{+\infty} \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha} dm(v + \omega) \end{aligned}$$

By change of variable, we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} |\gamma_{t,T}(y)|^{\alpha} dm(y - \frac{x}{M_T} + \omega) = \\ & \int_{-\infty}^{+\infty} |\gamma_{t,T}(y)|^{\alpha} h_0(y - \frac{x}{M_T} + \omega) dy \\ & \leq \sup h_0(y) \int_{-\infty}^{+\infty} |\gamma_{t,T}(y)|^{\alpha} dy \end{aligned}$$

from (13) we conclude that $I'_1 + I'_2 = O(\varepsilon^{\alpha/2})$. So we have

$$\begin{aligned} \psi_{t,T}(\omega - \frac{x}{M_T}) &= \int_{-\infty}^{+\infty} |A_t(v + \omega)|^{\alpha} \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha} dm(v + \omega) + O(\varepsilon^{\alpha/2}) \\ &= \int_{-\infty}^{+\infty} \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha} dH_t(v + \omega) + O(\varepsilon^{\alpha/2}) \\ &= \int_{-\infty}^{+\infty} \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha} h_t(v + \omega) dv + O(\varepsilon^{\alpha/2}). \end{aligned}$$

Inspired by papers [33]-[37], we assume that bandwidth of $h_t(v)$ is much larger than that of $|\gamma_{t,T}(v)|^{\alpha}$. This translates to: $|\gamma_{t,T}(v)|^{\alpha}$ is a "pseudo-function δ with respect to $h_t(v)$ in order $(\frac{B_{g_{\alpha,T}}}{B_X})$," which implies

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} \left| \gamma_{t,T}(v + \frac{x}{M_T}) \right|^{\alpha} h_t(v + \omega) dv - h_t(\omega) \right. \\ & \quad \left. - \frac{x}{M_T} \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^{\alpha} dv \right| < \frac{B_{g_{\alpha,T}}}{B_X} \leq \varepsilon \end{aligned}$$

so

$$\left| \psi_{t,T}(\omega - \frac{x}{M_T}) - h_t(\omega - \frac{x}{M_T}) \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^{\alpha} dv \right| \leq \text{const}(\varepsilon^{\alpha/2}),$$

hence

$$\begin{aligned} \psi_{t,T}(\omega - \frac{x}{M_T}) &= h_t(\omega - \frac{x}{M_T}) \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^{\alpha} dv \\ &+ O(\varepsilon^{\alpha/2}), \end{aligned}$$

since $h_t(\cdot)$ is continuous and from (13) we obtain

$$\lim_{T \rightarrow \infty} \psi_{t,T}(\omega - \frac{x}{M_T}) = h_t(\omega) + \left(\frac{\alpha}{\varepsilon^2} \right) \quad (33)$$

uniformly in x .

Then since

$$E[\hat{f}_{t,T}(\omega)] = \int_{-1}^{+1} W(x) \left[\psi_{t,T}(\omega - \frac{x}{M_T}) \right]^{p/\alpha} dx,$$

we get

$$\lim_{T \rightarrow \infty} E[\hat{f}_{t,T}(\omega)] = [h_t(\omega)]^{p/\alpha} + O(\varepsilon^{p/2}).$$

Showing now the consistency of the estimator of $\hat{f}_{t,T}(\omega)$. This means that the limit of the variance of $\hat{f}_{t,T}(\omega)$ goes to zero.

Theorem 9.

Let $-\pi < \omega < \pi$ and $0 < p < \frac{\alpha}{2}$, then we have

$$\lim_{T \rightarrow \infty} \text{var}[\hat{f}_{t,T}(\omega)] = O(\varepsilon^{\alpha})$$

Proof:

The variance can be writing as follows:

$$\text{var}[\hat{f}_{t,T}(\omega)] =$$

$$\begin{aligned} & \int \int_{-\infty}^{+\infty} W_T(\omega - u_1) W_T(\omega - u_2) \text{Cov}[\hat{f}_{t,T}(u_1), \hat{f}_{t,T}(u_2)] du_1 du_2 \end{aligned} \quad (34)$$

$$= \int \int_{-\infty}^{+\infty} (M_T)^2 W(M_T(\omega - u_1)) W(M_T(\omega - u_2)) \text{Cov}[\hat{f}_{t,T}(u_1), \hat{f}_{t,T}(u_2)] du_1 du_2$$

putting $x_i = M_T(\omega - u_i)$ $i = 1, 2$

$$\begin{aligned} \text{var}[\hat{f}_{t,T}(\omega)] &= \int_{-1}^{+1} \int_{-1}^{+1} W(x_1) W(x_2) \text{Cov} \left[\hat{f}_{t,T}(\omega - \frac{x_1}{M_T}), \hat{f}_{t,T}(\omega - \frac{x_2}{M_T}) \right] dx_1 dx_2 \end{aligned}$$

We write

$$\text{var}[\hat{f}_{t,T}(\omega)] = \int \int_{|x_1 - x_2| \leq \sigma_T} + \int \int_{|x_1 - x_2| > \sigma_T} = J_1 + J_2$$

where σ_T is a strictly positive real sequence which converges to zero. Using the Cauchy-Schwarz inequality, we increase J_1 by:

$$\begin{aligned} \text{var}[\hat{I}_{t,T}(\omega - \frac{x}{M_T})] &= V_{\alpha,p} \left[\psi_{t,T}(\omega - \frac{x}{M_T}) \right]^{\frac{2p}{\alpha}} \\ &\rightarrow V_{\alpha,p} [h_t(\omega)]^{\frac{2p}{\alpha}} \end{aligned}$$

uniformly in $|x| \leq 1$, we obtain

$$|J_1| \leq \text{const} [h_t(\omega)]^{\frac{2p}{\alpha}} \sigma_T. \quad (35)$$

On the other hand from (12) and (13) we obtain

$$\begin{aligned} E[\hat{I}_{t,T}(\omega - \frac{x_1}{M_T})] - \hat{I}_{t,T}(\omega - \frac{x_1}{M_T}) \\ = \\ \frac{1}{F_{p,\alpha}[C_\alpha]^{p/\alpha}} \int_{-\infty}^{+\infty} \text{Re} e^{-iy_1 I_{t,T}(\omega - \frac{x_1}{M_T})} e^{-|y_1|^\alpha C_\alpha \psi_{t,T}(\omega - \frac{x_1}{M_T})} dy_1. \end{aligned}$$

Using the fact that $\cos(x) = \frac{1}{2} \{e^{ix} + e^{-ix}\}$ and the proposition 2, we deduce that

$$\begin{aligned} \text{cov} \left[\hat{I}_{t,T}(\omega - \frac{x_1}{M_T}), \hat{I}_{t,T}(\omega - \frac{x_2}{M_T}) \right] = \\ \frac{1}{F_{p,\alpha}^2 [C_\alpha]^{2p/\alpha}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{|y_1 y_2|^{p+1}} \left\{ E \left[\prod_{j=1}^2 \cos(y_j I_{t,T}(\omega - \frac{x_j}{M_T})) \right] \right. \\ \left. - \exp - C_\alpha \sum_{j=1}^2 |y_j|^\alpha \psi_{t,T}(\omega - \frac{x_j}{M_T}) \right\} dy_1 dy_2. \end{aligned}$$

the equality $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$ gives

$$\begin{aligned} E \left[\prod_{j=1}^2 \cos(y_j I_{t,T}(\omega - \frac{x_j}{M_T})) \right] \\ = \frac{1}{2} \exp - C_\alpha \int \left| \sum_{j=1}^2 y_j \int_{t-T}^{t+T} g_{\alpha,T}(\lambda) \varphi_{t-\lambda}(v) \exp \left[-i \left\{ \omega - \frac{x_j}{M_T} \right\} \{t - \lambda\} \right] d\lambda \right|^\alpha dm(v) \end{aligned}$$

+exp

$$\begin{aligned} - C_\alpha \int \left| \sum_{j=1}^2 (-1)^{j+1} y_j \int_{t-T}^{t+T} g_{\alpha,T}(\lambda) \varphi_{t-\lambda}(v) \exp \left[-i \left\{ \omega - \frac{x_j}{M_T} \right\} \{t - \lambda\} \right] d\lambda \right|^\alpha dm(v). \end{aligned}$$

Putting $y_2 = -v_2$, we get

$$\begin{aligned} \text{cov} \left[\hat{I}_{t,T}(\omega - \frac{x_1}{M_T}), \hat{I}_{t,T}(\omega - \frac{x_2}{M_T}) \right] \\ = \frac{1}{F_{p,\alpha}^2 [C_\alpha]^{2p/\alpha}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \{e^{-a} - e^{-b}\} \frac{dy_1 dy_2}{|y_1 y_2|^{p+1}} \end{aligned}$$

where

$$\begin{aligned} a &= C_\alpha \int \left| \sum_{j=1}^2 y_j \int_{t-T}^{t+T} g_{\alpha,T}(\lambda) \varphi_{t-\lambda}(v) \exp \left[-i \left\{ \omega - \frac{x_j}{M_T} \right\} \{t - \lambda\} \right] d\lambda \right|^\alpha dm(v) \\ b &= C_\alpha \sum_{j=1}^2 |y_j|^\alpha \psi_{t,T} \left(\omega - \frac{x_j}{M_T} \right). \end{aligned}$$

Since a and b are positive we use the following inequality:

$$|e^{-t} - e^{-r}| \leq e^{-r} |t - r| e^{|t-r|}$$

Substituting a by t and b by r , we have

$$\begin{aligned} \left| \text{cov} \left[\hat{I}_{t,T} \left(\omega - \frac{x_1}{M_T} \right), \hat{I}_{t,T} \left(\omega - \frac{x_2}{M_T} \right) \right] \right| \\ \leq \frac{1}{F_{p,\alpha}^2 [C_\alpha]^{2p/\alpha}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |a - b| e^{|a-b|-b} \frac{dy_1 dy_2}{|y_1 y_2|^{p+1}}. \end{aligned}$$

Using again the following inequality,

$|x + y|^\alpha - |x|^\alpha - |y|^\alpha \leq 2|xy|^{\alpha/2}$ for $x, y \in \mathbb{R}$ and $\alpha \in]0, 2]$, we have

$$\begin{aligned} |a - b| \leq \\ 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \int \prod_{j=1}^2 \left| \int_{t-T}^{t+T} g_{\alpha,T}(\lambda) \varphi_{t-\lambda}(v) \exp \left[-i \left\{ \omega - \frac{x_j}{M_T} \right\} \{t - \lambda\} \right] d\lambda \right|^{\alpha/2} dm(v) \leq \\ 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \int \prod_{j=1}^2 \left| \int_{t-T}^{t+T} g_{\alpha,T}(\lambda) A_{t-\lambda}(v) \exp(iv\{t - \lambda\}) \exp \left[-i \left\{ \omega - \frac{x_j}{M_T} \right\} \{t - \lambda\} \right] d\lambda \right|^{\alpha/2} dm(v). \end{aligned}$$

Using (11), (24), (25) and the following inequality, $|x + y|^q \leq |x|^q + |y|^q$ for $x, y \in \mathbb{R}$ and $q \in]0, 1]$, we obtain

$$\begin{aligned} |a - b| \leq \\ 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \int_{-\infty}^{+\infty} \prod_{j=1}^2 \left[|A_t(v + \omega)|^{\frac{\alpha}{2}} \left| \gamma_{t,T}(v + \frac{x_j}{M_T}) \right|^{\frac{\alpha}{2}} + \varepsilon^{\frac{\alpha}{2}} \right] dm(v + \omega) \leq \\ 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \int_{-\infty}^{+\infty} |A_t(v + \omega)|^\alpha \left| \gamma_{t,T}(v + \frac{x_1}{M_T}) \right|^{\frac{\alpha}{2}} \left| \gamma_{t,T}(v + \frac{x_2}{M_T}) \right|^{\frac{\alpha}{2}} dm(v + \omega) \\ + 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \varepsilon^{\frac{\alpha}{2}} \int_{-\infty}^{+\infty} |A_t(v + \omega)|^{\frac{\alpha}{2}} \left| \gamma_{t,T}(v + \frac{x_1}{M_T}) \right|^{\frac{\alpha}{2}} dm(v + \omega) \end{aligned}$$

$$\begin{aligned}
 & + 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \varepsilon^{\frac{\alpha}{2}} \int_{-\infty}^{+\infty} |A_t(v + \omega)|^{\frac{\alpha}{2}} \left| \gamma_{t,T}(v + \frac{x_2}{M_T}) \right|^{\frac{\alpha}{2}} dm(v + \omega) \\
 & + 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \varepsilon^\alpha \int_{-\infty}^{+\infty} dm(v + \omega)
 \end{aligned}$$

Now we will assume that $g_{\alpha,T}(\lambda)$ satisfies

$$\int_{-\infty}^{+\infty} |g_{\alpha,T}(\lambda)| d\lambda \leq G_\alpha < \infty \quad \text{for all } T,$$

where G_α is not dependent to T .

The the Fourier transform of $g_{\alpha,T}(\lambda)$, $\gamma_{t,T}(v)$, is called “transfer function”. If the precedent inequality holds this ensures that $\gamma_{t,T}(v)$ is bounded independently of T . Since $|A_{t-\lambda-u_j}(v)|$ is bounded by 1 then

$$\begin{aligned}
 |a - b| & \leq 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \sup(h_0) (G_\alpha)^{\frac{\alpha}{2}} \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^{\frac{\alpha}{2}} dv \\
 & + 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \varepsilon^{\frac{\alpha}{2}} \sup(h_0) \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^{\frac{\alpha}{2}} dv \\
 & + 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \varepsilon^\alpha \sup(h_0) \int_{-\infty}^{+\infty} |\gamma_{t,T}(v)|^{\frac{\alpha}{2}} dv \\
 & + 2C_\alpha |y_1 y_2|^{\frac{\alpha}{2}} \varepsilon^\alpha \int_{-\infty}^{+\infty} dm(v + \omega)
 \end{aligned}$$

Putting $L_T = \int_{-\infty}^{+\infty} |\gamma_{t,T}(v')|^{\frac{\alpha}{2}} dv'$, $C = \int_{-\infty}^{+\infty} dm(v + \omega)$ and the use of the inequality $2|st|^{\alpha/2} \leq |s|^\alpha + |t|^\alpha$ gives

$$\begin{aligned}
 |a - b| - b & \leq -C_\alpha \sum_{j=1}^2 |y_j|^\alpha \left\{ \psi_{T,t}(\omega - \frac{x_j}{M_T}) \right. \\
 & \left. - \left((G_\alpha)^{\frac{\alpha}{2}} \sup(h_0) L_T + 2\varepsilon^{\frac{\alpha}{2}} \sup(h_0) L_T + \varepsilon^\alpha C \right) \right\}
 \end{aligned}$$

which imply that

$$\begin{aligned}
 & \left| cov \left[\hat{f}_{t,T}(\omega - \frac{x_1}{M_T}), \hat{f}_{t,T}(\omega - \frac{x_2}{M_T}) \right] \right| \leq \\
 & \frac{1}{F_{p,\alpha}^2 [C_\alpha]^{2p/\alpha}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} 2C_\alpha \left((G_\alpha)^{\frac{\alpha}{2}} \sup(h_0) L_T + 2\varepsilon^{\frac{\alpha}{2}} \sup(h_0) L_T + \varepsilon^\alpha C \right) * \\
 & \exp \left[-C_\alpha \sum_{j=1}^2 |y_j|^\alpha \left(\psi_{T,t}(\omega - \frac{x_j}{M_T}) - (G_\alpha)^{\frac{\alpha}{2}} \sup(h_0) L_T \right. \right. \\
 & \quad \left. \left. - 2\varepsilon^{\frac{\alpha}{2}} \sup(h_0) L_T - \varepsilon^\alpha C \right) \right] \frac{dy_1 dy_2}{|y_1 y_2|^{p+1-\frac{\alpha}{2}}} \\
 & \leq \frac{2C_\alpha \left((G_\alpha)^{\frac{\alpha}{2}} \sup(h_0) L_T + 2\varepsilon^{\frac{\alpha}{2}} \sup(h_0) L_T + \varepsilon^\alpha C \right)}{F_{p,\alpha}^2 [C_\alpha]^{\frac{2p}{\alpha}}} \\
 & \prod_{j=1}^2 \int_{-\infty}^{+\infty} \exp \left[-C_\alpha |y_j|^\alpha \Delta_{T,t}(\omega - \frac{x_j}{M_T}) \right] \frac{dy_j}{|y_j|^{p+1-\frac{\alpha}{2}}},
 \end{aligned}$$

where

$$\begin{aligned}
 \Delta_{T,t} \left(\omega - \frac{x_j}{M_T} \right) & = \\
 \psi_{T,t}(\omega - \frac{x_j}{M_T}) - (G_\alpha)^{\frac{\alpha}{2}} \sup(h_0) L_T - 2\varepsilon^{\frac{\alpha}{2}} \sup(h_0) L_T - \varepsilon^\alpha C & \quad (36)
 \end{aligned}$$

Putting $z_j = y_j [\Delta_{T,t}(\omega - \frac{x_j}{M_T})]^{\frac{1}{\alpha}}$, we obtain

$$\begin{aligned}
 & \left| cov \left[\hat{f}_{t,T} \left(\omega - \frac{x_1}{M_T} \right), \hat{f}_{t,T} \left(\omega - \frac{x_2}{M_T} \right) \right] \right| \leq \\
 & \frac{2C_\alpha \left((G_\alpha)^{\frac{\alpha}{2}} \sup(h_0) L_T + 2\varepsilon^{\frac{\alpha}{2}} \sup(h_0) L_T + \varepsilon^\alpha C \right)}{F_{p,\alpha}^2 [C_\alpha]^{2p/\alpha}} \\
 & \times \prod_{j=1}^2 \int_{-\infty}^{+\infty} \frac{\exp[-C_\alpha |z_j|^\alpha]}{|z_j|^{p+1-\frac{\alpha}{2}}} \frac{\left\{ [\Delta_{T,t}(\omega - \frac{x_j}{M_T})]^{\frac{1}{\alpha}} \right\}^{p+1-\frac{\alpha}{2}}}{[\Delta_{T,t}(\omega - \frac{x_j}{M_T})]^{\frac{1}{\alpha}}} dz_j \leq \\
 & \frac{2C_\alpha \left((G_\alpha)^{\frac{\alpha}{2}} \sup(h_0) L_T + 2\varepsilon^{\frac{\alpha}{2}} \sup(h_0) L_T + \varepsilon^\alpha C \right) F_{p,\alpha}^{-2} [C_\alpha]^{-\frac{2p}{\alpha}}}{[\Delta_{T,t}(\omega - \frac{x_1}{M_T}) \Delta_{T,t}(\omega - \frac{x_2}{M_T})]^{\frac{1}{2} \frac{p}{\alpha}}} \times \\
 & \prod_{j=1}^2 \int_{-\infty}^{+\infty} \frac{\exp(-C_\alpha |z_j|^\alpha)}{|z_j|^{p+1-\frac{\alpha}{2}}} dz_j.
 \end{aligned}$$

Since $\lim_{T \rightarrow \infty} \psi_{T,t}(\omega - \frac{x_j}{M_T}) = h_t(\omega) + O(\varepsilon^{\alpha/2})$ and using the condition (13), we have

$$\lim_{T \rightarrow \infty} \Delta_{T,t}(\omega - \frac{x_j}{M_T}) = h_t(\omega) + O(\varepsilon^{\alpha/2}).$$

Since the integral $\int_{-\infty}^{+\infty} \frac{\exp(-C_\alpha |z_j|^\alpha)}{|z_j|^{p+1-\frac{\alpha}{2}}} dz_j$ is finite, we obtain

$$\lim_{T \rightarrow \infty} cov \left[\hat{f}_{t,T} \left(\omega - \frac{x_1}{M_T} \right), \hat{f}_{t,T} \left(\omega - \frac{x_2}{M_T} \right) \right] = O(\varepsilon^\alpha) \text{ uniformly in } x_1, x_2.$$

Thus

$$J_2 = O(\varepsilon^\alpha)$$

From (29), we obtain that

$$\lim_{T \rightarrow \infty} var[\hat{f}_{t,T}(\omega)] = O(\varepsilon^\alpha)$$

Let ω belong to $]-\pi, \pi[$ then we have

$$\lim_{T \rightarrow \infty} E[\hat{f}_{t,T}(\omega) - (h_t(\omega))^{p/\alpha}]^2 = O(\varepsilon^p)$$

In order to apply the results which had seen, we must write

$$\begin{aligned}
 E[\hat{f}_{t,T}(\omega) - (h_t(\omega))^{\frac{p}{\alpha}}]^2 & = \\
 & = [E[\hat{f}_{t,T}(\omega)] - (h_t(\omega))^{p/\alpha}]^2 \\
 & + var[\hat{f}_{t,T}(\omega)].
 \end{aligned}$$

From proposition 1, theorems 5 and 8, we obtain

$$\lim_{T \rightarrow \infty} E[\hat{f}_{t,T}(\omega) - (h_t(\omega))^{p/\alpha}]^2 = O(\varepsilon^p).$$

Let ω belong to $]-\pi, \pi[$, then $\{\hat{f}_{t,T}(\omega)\}^{\frac{\alpha}{p}}$ converge in probability to $h_t(\omega)$.

Using the following inequality: $|y^q - x^q| \leq \frac{q}{2} |y - x| (y^{q-1} + x^{q-1})$, for all $x, y \in R^+$ and $q > 2$, we obtain

$$\begin{aligned}
 & \left| \{\hat{f}_{t,T}(\omega)\}^{\frac{\alpha}{p}} - h_t(\omega) \right| \leq \\
 & \frac{\alpha}{2p} \left| \hat{f}_{t,T}(\omega) - h_t(\omega)^{\frac{p}{\alpha}} \right| \times \left| \{\hat{f}_{t,T}(\omega)\}^{\frac{\alpha}{p}-1} + \{h_t(\omega)^{\frac{p}{\alpha}}\}^{\frac{\alpha}{p}-1} \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\alpha}{2p} \left| \hat{f}_{t,T}(\omega) - h_t(\omega)^{\frac{p}{\alpha}} \right| \times \\
 &\left| \left\{ \hat{f}_{t,T}(\omega) \right\}^{\frac{\alpha}{p}-1} - \left\{ h_t(\omega)^{\frac{p}{\alpha}} \right\}^{\frac{\alpha}{p}-1} + 2 \left\{ h_t(\omega)^{\frac{p}{\alpha}} \right\}^{\frac{\alpha}{p}-1} \right| \\
 &\leq \frac{\alpha}{2p} \left| \hat{f}_{t,T}(\omega) - h_t(\omega)^{\frac{p}{\alpha}} \right| \times \\
 &\left| \left\{ \hat{f}_{t,T}(\omega) \right\}^{\frac{\alpha}{p}-1} - \left\{ h_t(\omega)^{\frac{p}{\alpha}} \right\}^{\frac{\alpha}{p}-1} \right| \\
 &+ \frac{\alpha}{p} \left| \hat{f}_{t,T}(\omega) - h_t(\omega)^{\frac{p}{\alpha}} \right| \left| \left\{ h_t(\omega)^{\frac{p}{\alpha}} \right\}^{\frac{\alpha}{p}-1} \right|.
 \end{aligned}$$

Since $\hat{f}_{t,T}(\omega)$ converges to $h_t(\omega)^{\frac{p}{\alpha}}$ in L^2 we deduct that $\hat{f}_{t,T}(\omega)$ converge to $h_t(\omega)^{\frac{p}{\alpha}}$ in probability. The function $u \rightarrow u^{q-1}$ is convex, so $(\hat{f}_{t,T}(\omega))^{q-1}$ converges to $(h_t(\omega)^{\frac{p}{\alpha}})^{q-1}$ in probability. Using the fact that h_t is bounded we have

$$\begin{aligned}
 &P \left[\left| \left\{ \hat{f}_{t,T}(\omega) \right\}^{\frac{\alpha}{p}} - h_t(\omega)^{\frac{p}{\alpha}} \right| > \varepsilon' \right] \leq \\
 &P \left[\left| \frac{\alpha}{2p} \left| \hat{f}_{t,T}(\omega) - h_t(\omega)^{\frac{p}{\alpha}} \right| > \left(\frac{\varepsilon'}{2} \right)^{\frac{1}{2}} \right] + \\
 &P \left[\left| \left\{ \hat{f}_{t,T}(\omega) \right\}^{\frac{\alpha}{p}-1} - \left\{ h_t(\omega)^{\frac{p}{\alpha}} \right\}^{\frac{\alpha}{p}-1} \right| > \left(\frac{\varepsilon'}{2} \right)^{\frac{1}{2}} \right] \\
 &P \left[\left| \frac{\alpha}{p} \left| \hat{f}_{t,T}(\omega) - h_t(\omega)^{\frac{p}{\alpha}} \right| \sup_{\omega} \left\{ h_t(\omega)^{\frac{p}{\alpha}} \right\}^{\frac{\alpha}{p}-1} \right| > \frac{\varepsilon'}{2} \right]
 \end{aligned}$$

for all $\varepsilon' > 0$, which implies the result of this theorem.

IV. CONCLUSION

This article focuses on the estimation of the evolving spectral density when the process studied has infinite variance, in particular when it is alpha symmetric stable. The applications of this kind of process are numerous, in particular for random phenomena whose observations form a large fluctuation. For example:

- In the study of music, to isolate the sound of an instrument among others. The audio signal would be tainted by background noise with a very large fluctuation.
- The movement of sea waves constitutes a phenomenon whose variance is very large. The distribution of its energy could be estimated by an evolving spectral density.

This work can be completed by the study of the optimization of the smoothing parameter which will allow an optimal improvement of the convergence rate of the proposed estimator.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

AUTHOR CONTRIBUTIONS

This work is carried out by both authors with equal contribution at all stages: research and data analysis and writing. All authors had approved the final version.

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