The Choice of the Smoothing Parameter for Alpha Stable Signals

Rachid Sabre
Biogéosciences (UMR CNRS/uB 6282), University of Burgundy, Agrosup, 26, Bd Docteur Petitjean, Dijon, France
Email: r.sabre@agrosupdijon.fr

***Abstract*—In this work we consider the class of symmetric alpha stable processes which are a particular family of processes with infinite energy. These processes used in modeling the random signals with indefinitely growing variance. The spectral density estimator of such signals is given in the literature by smoothing the periodogram by a spectral window. Thus, the estimator depends on the width of the spectral window considered as a smoothing parameter. The choice of this parameter plays an important role since the rate of convergence of the estimator is a function of this parameter. The objective of this paper is to propose a method giving the optimal parameter based on the cross validation technique (minimization of MISE: Mean Integrate Square of Error). We establish a criterion function and we prove that the mean of this criterion converges to MISE. Thus, we show that the value minimizing this criterion is the optimal smoothing parameter. The rate of convergence of the estimator has been studied in order to prove that the smoothing parameter obtained by this method gives the fastest convergence of the estimator towards the spectral density.**

***Index Terms*—alpha stable, cross validation, spectral density, spectral window**

I. Introduction

In this paper, a class of symmetric alpha stable signals has been considered. It is a particular family of processes with infinite energy. Theory of these processes have been covered in a numerous papers including [1]-[6] to name a few.

Symmetric alpha processes are considerably accurate model for many phenomenon in several fields such as: physics, biology, electronic and electric, hydrology, economies, communications and radar applications, (see [7]-[17]). In this work, a symmetric stable harmonizable process is precisely discussed $Z = \{Z_n : n \in Z\}$ Alternatively Zhas the integral representation:

$$Z_n = \int_{-\pi}^{\pi} \exp[i(n\lambda)] d\xi(\lambda)$$

where $1 < \alpha < 2$ and ξ is a complex valued symmetric α-stable random measure on R with independent and isotropic increments. The measure defined by $m(A) = |\xi(A)|_\alpha^\alpha$ (see [4]) is called the control measure or spectral measure. Suppose that this measure is absolutely continuous with respect to Lebesgue measure: $md(x) = \phi(x)dx$. The functionϕ is called the spectral density. The spectral density function was already estimated when the time of the process is continuous by [4], when the time of the process is discrete by [18] and when the time of the process is p-adic by [19].

The estimators of the spectral density proposed in literature use a smoothing parameter that satisfies certain conditions. The rate of convergence depends on this smoothing parameter. The choice of this parameter becomes problematic insofar as there are several parameters that satisfy these conditions. Few works deal with the choice of this parameter. The contribution of this work consists in giving, from the data, a criterion for choosing the optimal smoothing parameters allowing the estimators to converge as quickly as possible towards the spectral density. The originality of this work is the use of cross-validation to set up a criterion to choose the smoothing parameter minimizing the quadratic error in the spectral estimates for alpha stable processes. Cross validation has proven its worth in several situations, but this work uses it for the first time in the estimations of the spectral density of stable alpha processes.

This paper is organized as follows: section I is reserved to recall the periodogram and its smoothing to obtain a spectral density estimator. In section II we give the cross validation criterion. Section III show that criterion give the optimum parameter. The section IV is reserved for the numerical studies and simulation

II. Periodogram and Its Proprieties

This paper considers a (SαS) process where its spectral representation is

$$Z_n = \int_{-\pi}^{\pi} e^{in\lambda} d\xi(\lambda) \quad (1)$$

where ξ is a isotropic symmetric α -stable with independent increments

The measure defined by: $\mu(]s,t]) = |\xi(t) - \xi(s)|_\alpha^\alpha$ is Lebesgue-Stiel measure called the spectral measure (see [1] and [3]). When $\mu$is absolutely continuous $d\mu(x) = f(x)dx$, the function f is called the spectral density of the process$Z$.

As in [18], [20] and [21], we give the definition of the Jackson polynomial kernel:

Let $Z_1, \dots, Z_N$ observations of the process Z : $(Z_n)_{0 \le n \le N-1}$, where N satisfies:

$$N - 1 = 2k(n - 1) \text{ with } n \in N\ k \in N \cup \{1/2\}$$

If =1/2 then n=$2n_1$−1, n_1∈N.

Manuscript received January 14, 2020; revised May 14, 2020.

The Jackson's polynomial kernel is defined by:

$$|H_N(\lambda)|^\alpha = \left|A_N H^{(N)}(\lambda)\right|^\alpha \quad (2)$$

where

$$H^{(N)}(\lambda) = \frac{1}{q_{k,n}}\left(\frac{\sin\left(\frac{n\lambda}{2}\right)}{\sin\left(\frac{\lambda}{2}\right)}\right)^{2k}$$

$$with\ q_{k,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\sin\left(\frac{n\lambda}{2}\right)}{\sin\left(\frac{\lambda}{2}\right)}\right)^{2k} d\lambda.$$

and $A_N = (B_{\alpha,N})^{-\frac{1}{\alpha}}$ with $B_{\alpha,N} = \int_{-\pi}^{\pi}|H^{(N)}(\lambda)|^\alpha d\lambda$.

We give the following lemmas proved in [18].

Lemma 1.

There is a nonnegative function h_k such as:

$$H^{(N)}(\lambda) = \sum_{m=-k(n-1)}^{k(n-1)} h_k\left(\frac{m}{n}\right)\cos(m\lambda)$$

Let

$$B'_{\alpha,N} = \int_{-\pi}^{\pi}\left|\frac{\sin\left(\frac{n\lambda}{2}\right)}{\sin\left(\frac{\lambda}{2}\right)}\right|^{2k\alpha} d\lambda$$

and $J_{N,\alpha} = \int_{-\pi}^{\pi}|u|^\gamma|H_N(\lambda)|^\alpha d\lambda$, where $\gamma \in]0,2]$.

Then $B'_{\alpha,N}\begin{cases}\geq 2\pi\left(\frac{2}{\pi}\right)^{2k\alpha} n^{2k\alpha-1} \text{if } 0 < \alpha < 2\\ \leq \frac{4\pi k\alpha}{2k\alpha-1} n^{2k\alpha-1} \text{if } \frac{1}{2k} < \alpha < 2\end{cases}$

and $J_{N,\alpha} \leq \begin{cases}\frac{\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma-2k\alpha+1)}\frac{1}{n^{2k\alpha-1}} \text{ if } \frac{1}{2k} < \alpha < \frac{\gamma+1}{2k},\\ \frac{2k\alpha\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma+1)(2k\alpha-\gamma-1)}\frac{1}{n^\gamma} \text{ if } \frac{\gamma+1}{2k} < \alpha < 2.\end{cases}$

The following lemma is proved in [1].

Lemma 2.

If ξ is a (S. α .S) process with independent and isotropic increments, then

$$E\left[\exp\left(iRe\left[i\int_{-\pi}^{\pi} f(u)d\xi(u)\right]\right)\right] = \exp\left(-C_\alpha\int_{-\pi}^{\pi}|f(u)|^\alpha d\mu(u)\right)$$

where $C_\alpha = (2\pi)^{-1}\int_{-\pi}^{\pi}|cos(\theta)|^\alpha d\mu(u)$.

In this paper, we propose the following periodogram defined by

$$d_N(\lambda) = A_N\sum_{n'=-k(n-1)}^{k((n-1)} h_k(n'/n)\left(e^{-in'\lambda}\right)X(n'+k(n'-1)).$$

Using the lemma 2 we show that

$$E\exp(irRed_N(\lambda)) = \exp(-C_\alpha|r|^\alpha\psi_N(\lambda))$$

where

$$\psi_N(\lambda) = \int_{-\pi}^{\pi}\left|A_N\sum_{n'=-k(n-1)}^{k((n-1)} h_k(n'/n)e^{-in'\lambda}e^{n'u}\right|^\alpha \phi(u)du$$

Thus

$$\psi_N(\lambda) = \int_{-\pi}^{\pi}|H_N(\lambda-u) - H_N(u)|^\alpha\phi(u)du)$$

We modify this periodogram by taking the power p, $0 < p < \frac{\alpha}{2}$, and multiplying by a normalization constant:

$$I_N(\lambda) = C_{(p,\alpha)}|d_N(\lambda)|^p$$

The normalization constant is given by

$$C_{(p,\alpha)} = \frac{D_p}{F_{p,\alpha}C_\alpha^{p/\alpha}}$$

where $D_p = \int\frac{1-cos(u)}{|u|^{1+p}}du$ and $F_{p,\alpha} = \int\frac{1-e^{-|u|^\alpha}}{|u|^{1+p}}du$

As in [3] and [18], we show that

$$EI_N(\lambda) = (\psi_N(\lambda))^{\frac{p}{\alpha}}$$

and $Var(I_N(\lambda)) = V_{\alpha,p}\psi_N(\lambda)^{\frac{2p}{\alpha}}$

III. Smoothed Periodogram

In order to give an unbiased consistent estimate of ϕ, we smooth $\hat{I}_N$by a the following spectral window:

$$f_N(\lambda) = \int_{-\pi}^{\pi}W_N(\lambda-u)I_N(u)du$$

where the spectral window is defined by $W_N(x) = M_N W(M_N x)$ where W is a nonnegative even continuous function vanishing for $|x| > 1$ with $\int_{-1}^{1}W(x)dx = 1$ and M_N is sequence converging to infinity such that $\frac{M_N}{N} \to 0$.

As in [20] for giving the best rate of convergence of this estimator, we introduce on ϕ two hypothesis (h_1) and (h_2) called regularity hypothesis:

(h_1): $|\phi(\lambda-u) - \phi(\lambda)| \leq C_1|u|^\gamma$ where $0 < \gamma \leq 1$

(h_2) : $|\phi(\lambda-u) - \phi(\lambda) - u\phi'(\lambda)| \leq C_2|u|^\gamma$ where $1 \leq \gamma \leq 2$ C_1 and C_2 being nonnegative constants.

Note by $f(\lambda) = \phi(\lambda)^{\frac{p}{\alpha}}$

Theorem 1.

Let λ a real number. Then

(i) $f_N(\lambda)$ is an asymptotically unbiased estimator of the $f(\lambda)$

(ii) Choosing k so large that $+1 < 2k\alpha n$ we have

$$Ef_N(\lambda) - f(\lambda) = \begin{cases}O\left(\frac{1}{M_N{}^\gamma}\right) \text{ if } \phi \text{ satisfies}(h_1)\\ O\left(\frac{1}{M_N}\right) \text{ if } \phi \text{ satisfies}(h_2)\end{cases}$$

(iii) $Var(f_N(\lambda))$ converges to zero.

Vi) If ϕ satisfies (h_1) or (h_2) and $M_N{}^2/n$ converges to zero then $Var(f_N(\lambda)) = O\left(\frac{M_N{}^4}{n^2}\right)$

The proof of this theorem is inspired by the demonstration used in [18].

Theorem 2.

Let λ *a real number such that* $\phi(\lambda) > 0$. *Then* $(f_N(\lambda))^{\frac{\alpha}{p}}$ *converges in probability to* $\phi(\lambda)$.

Proof

We show that $f_N(\lambda)$ converges in mean quadratic to $f(\lambda)$. Indeed

$$E\left|f_N(\lambda) - \phi(\lambda)^{\frac{p}{\alpha}}\right|^2 = \left(Ef_N(\lambda) - \phi(\lambda)^{\frac{p}{\alpha}}\right)^2 + Var(f_N(\lambda)).$$

Then from theorem 1, $E\left|f_N(\lambda) - \phi(\lambda)^{\frac{p}{\alpha}}\right|^2$ converges to zero. Thus $(f_N(\lambda))^{\frac{\alpha}{p}}$ *converges in probability to* $(\phi(\lambda))^{\frac{\alpha}{p}} = f(\lambda)$.

It is clear that the choice of M_N plays an important role since the convergence speeds depend on this smoothing parameter. The paper [22] gave a criterion of choice of h in the one-dimensional case, they were restricted to the parametric case. The objective of this work is to give a criterion for the selection of these parameters by non-parametric methods. Let's note by $h = \frac{1}{M_N}$ the width of the spectral window. We are therefore looking for a criterion $CV(h)$ allowing us to select h minimize the mean integrated square error (MISE), where

$$\mathrm{MISE(h)} = \int \mathrm{E}[\mathrm{f_N(x)} - \mathrm{f(x)}]^2 \rho(\mathrm{x})d\mathrm{x} \qquad (3)$$

ρ being a weight function that is assumed to be known and null outside of $[0,2\pi]$.

Although $MISE(\mathrm{h})$ it is a good measure of the quality of f_{N}, it can not help us to choose h, since it depends on the unknown function f. We will therefore try to estimate it. For this, we adopt the method of cross validation that has been proposed by [22]. Indeed, consider the integrated square error (ISE) defined by:

$$ISE(\mathrm{h}) = \int [f_N(\mathrm{x}) - f(\mathrm{x})]^2 \rho(\mathrm{x})d\mathrm{x} = A - 2C + B$$

where $A = \int_0^{2\pi} f_N^2(\mathrm{x})\rho(\mathrm{x})d\mathrm{x}$

$$C = \int_0^{2\pi} f_N(\mathrm{x})f(\mathrm{x})\rho(\mathrm{x})d\mathrm{x}$$

$$B = \int_0^{2\pi} f^2(\mathrm{x})\rho(\mathrm{x})d\mathrm{x}.$$

Since B is Independent of h, to choice h, minimizing $ISE(h)$ is to choose h minimiszing $A - 2C$. We can calculate the term A since we know f_N, whereas, in the term C, since f is unknown. We proceed by the principle of "leave-out- I".

IV. Construction of the Cross Validated Estimator

In this section, we will define the estimator and give some results in the form of a proposition or theorem.

Let $j \in \{0,1,\dots,n-1\}$. The construction of "leave-out- I" consiste of find an estimator $f_N^j(\omega_j)$ that replace $f_N(\omega_j)$ in the expression of C and such that $I_N(\omega_j)$ and $f_N^j(\omega_j)$ are asymptotically independent. Thus, we can estimate C by:

$$\frac{1}{\overline{N}}\sum_{j=1}^{\overline{N}} f_N^j(\omega_j)I_N(\omega_j)\rho(\omega_j)$$

$\frac{1}{\overline{N}}\sum_{j=1}^{\overline{N}} f_N^j(\omega_j)I_N(\omega_j)\rho(\omega_j)$ where $\omega_j = \frac{2\pi j}{N}$,

$\overline{N} = \left[\frac{N-1}{2}\right]$, $f_N^j(\mathrm{x}) = \int_0^{2\pi} I_N^j(\mathrm{u})W_N(\mathrm{x}-\mathrm{u})d\mathrm{u}$ with

$I_N^j(u) = I_N(\mathrm{u})$ si $\mathrm{u} \notin A_j$

$I_N^j(u) = \theta_1(\mathrm{u})I_N(\omega_{j+1}) + \theta_2(\mathrm{u})I_N(\omega_{j-1})$ sinon

$A_j =]\omega_{j-1}, \omega_{j+1}[$,

where

$$\theta_1(\mathrm{u}) = \alpha\ ;\ \theta_2(u) = (1-\alpha) \text{ with } \alpha = \frac{\mathrm{u}-\omega_{j+1}}{\omega_{j-1}-\omega_{j+1}}.$$

The following proposition shows that f_N^j and f_N have asymptotically same limit.

Proposition 1. *We obtain that*

$$E\left[f_N^j(\mathrm{x}) - f_N(\mathrm{x})\right] = O\left(\frac{1}{N}\right).$$

From this result, we establish our criterion, noted CV "cross validation", defined by:

$CV(h) = CV_1(\mathrm{h}) + \int_0^{2\pi} f^2(\mathrm{u})\rho(\mathrm{u})d\mathrm{u}$ where

$CV_1(\mathrm{h}) = \int_0^{2\pi} f_{\mathrm{N}}^2(\mathrm{u})\rho(\mathrm{u})d\mathrm{u} - \frac{2}{\mathrm{N}}\sum_{j=1}^{\overline{N}} f_{\mathrm{N}}^j(\omega_j)\mathrm{I_N}(\omega_j)\rho(\omega_j)$

We choose the optimal widths of spectral window $\hat{h}$ the value of h minimizing the criterion $CV(h)$:

$$\hat{\mathrm{h}} = \underset{\mathrm{h}}{\mathrm{argmin}}\mathrm{CV(h)} = \underset{\mathrm{h}}{\mathrm{argmin}}\mathrm{CV_1(h)} \qquad (4)$$

Subsequently, to facilitate writing and without losing generality, we will take $\rho(\mathrm{u}) = \frac{1}{2\pi}$ sur $[0,2\pi]$ and null outside.

V. Optimality of the Criterion

In this section, we establish results similar to those given by [23] and [24], concerning the estimation of the intensities of a punctual process. It is to show that, on average, when N are large enough, the criterion $CV(h)$is approximately equal to the integrated quadratic error $ISE(h)$and that the variance of $CV(h)$ is asymptotically zero. This allows us to confirm that the parameters $\hat{h}$ minimizing the criterion $CV(h)$ are close to those that minimize the integral squared error (ISE) when N are large enough. These results are stated in the following theorem:

Theorem 1. We have

$$|E\{CV(h) - ISE(h)\}| = O\left(\frac{1}{N}\right).$$

$$\mathrm{var}\{CV(h)\} = O\left(\frac{1}{Nh}\right)$$

Thus, since

$$E\{[CV(h) - MISE(h)]^2\} =$$
$$\text{var}\{CV(h)\} + [E\{CV(h) - MISE(h)\}]^2 \ = O\left(\frac{1}{Nh}\right).$$

The widths of the spectral window $\hat{h}$ obtained by cross validation, defined in (6), are asymptotically optimal, i.e. the integrated square error at $\hat{h}$ converges in probability to the small integrated square error:

Theorem 2. The width of the spectral window $\hat{h}$ obtained by cross validation are asymptotically optimal:

$\frac{ISE(\hat{h})}{ISE(\hat{\hat{h}}_1)} \to 1$ en probabilite , where

$$\hat{h} = \underset{h}{\text{argmin}} CV(h) \text{ and } (\hat{\hat{h}}_1) = \underset{h}{\text{argmin}} ISE(h).$$

To show this result we use the similar technique used in [25] and [26].

VI. Sumilation

The proposed estimator can be applied to concrete situations. For example, the transmission signal for the future generation of wireless and radio communication systems where multipath propagation leads to a significant degradation of the quality of the transmission. To solve this problem, the paper [11] proposed an arrival time model based on Poisson distributions. The paper [27] provided a model based on stable alpha distributions. The sum of arrival times modeled by independent and isotropic Poisson distributions can be represented by a stable harmonizable process like that given in (1), see [28].

Throughout this section, we give the simulation of the studied process:

$$Z_n = \int_{-\pi}^{\pi} e^{in\lambda} d\xi(\lambda)$$

where $1 < \alpha < 2$ and ξ is a complex symmetric α -stable measure on R with independent and isotropic increments and with control measure m such that $mdx = \phi(x)dx$.

In order to achieve this, we use the series representation defined in [28]. Therefore, the process Z given in (8) can be expressed as follows:

$Z_n = C_\alpha\left(\int \phi(x)dx\right)^{1\alpha} \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1\alpha} e^{inV_j} e^{i\theta_j}$ where

- ε_j is a sequence of i.i.d. random variables such as $P[\varepsilon_j = 0] = P[\varepsilon_j = 1] = 1/2$,
- Γ_k is a sequence of arrival times of Poisson process,
- V_j is a sequence of i.i.d. random variables independent of ε_k and of Γ_k having the same distribution of control measure m , which has probability density ϕ
- θ_j are independent random variables, having the uniform distribution on $[-\pi, \pi]$, independent of ε_j, Γ_j and V_j.

To generate N values ($N = 5001$) of the process Z_n, we use the following steps:

- generate 5000 values of ε_j
- generate 5000 values of Γ_j
- generate 5000 values of V_j
- generate 5000 values of θ_j

Then we calculate for all $0 \le n \le N$:

$$Z_n = C_\alpha\left(\int \phi(x)dx\right)^{\frac{1}{\alpha}} \sum_{j=1}^{2000} \varepsilon_j \Gamma_j^{-1\alpha} e^{inV_j} e^{i\theta_j}$$

where the spectral density is chosen as $f(x) = (\phi(x))^{\frac{p}{\alpha}} = |x|^2 e^{-|x|}$ for $x \in [-\pi, \pi]$ and $f(x) = 0$ otherwise and $\alpha = 1{,}7$.

We calculate the function $CV_1(\text{h})$ for $h \in [0{,}1]$. The curve of CV_1 is plotted on [0,1] in Fig. 1.

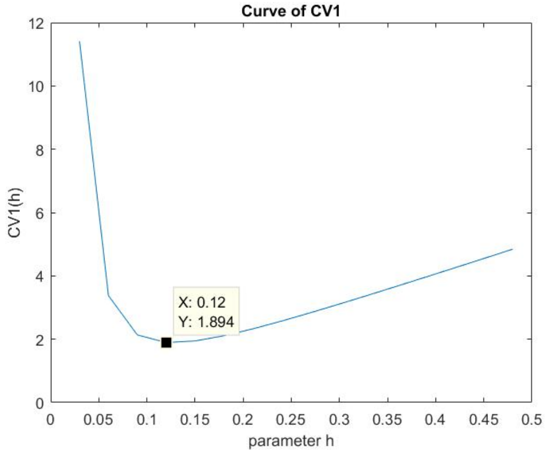

Figure 1. Curve of CV_1.

From the curve of CV1 we determine the value of h where CV1 is minimum: $\hat{h} = \underset{h}{\text{argmin}} CV_1(h) = 0.12$. Thus we deducting the optimal value of $M_N = \frac{1}{\hat{h}}$.

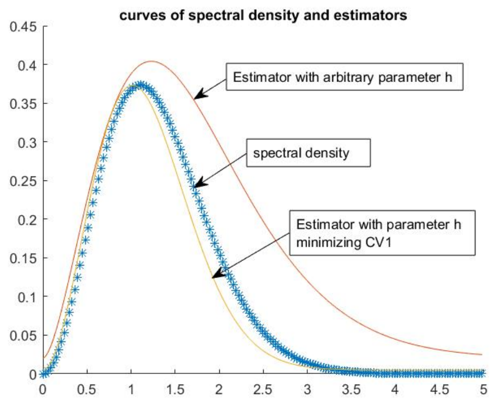

Figure 2. Curves of spractral density, its estimators.

Fig. 2 gives the curves of spectra density *f*, estimator of f with arbitrary parameter h and estimator of f with arbitrary parameter $\hat{h}$. We find that the estimator with the parameter $\hat{h}$ gives a better estimate.

VII. Conclusion

We give an estimator of the constant additive error in spectral representation of (S α S) process. This work could be applied to several cases when processes have an infinite variance and the observation of these processes are perturbed by a constant noise. For example:

- the decomposition of audio signals with background noise by separating the different musical instruments.
- the denoising of a degraded historical record. The signal is considered infinitely variable.

Conflict of Interest

The author declare no conflict of interest.

Acknowledgment

The author would like to thank the anonymous referees for their interest in this paper and their valuable comments and suggestions.

References

[1] S. Cambanis, "Complex symmetric stable variables and processes," *Contributions to Statistics: Essays in Honour of Norman L. Johnson*, New York: North-Holland, 1983, pp. 63-79.

[2] S. Cambanis and M. Maejima, "Two classes of self-similar stable processes with stationary increments", *Stochastic Process. Appl.* vol. 32, pp. 305-329, 1989.

[3] E. Masry and S. Cambanis, "Spectral density estimation for stationary stable processes," *Stochastic Processes and Their Applications*, vol. 18, pp. 1-31, 1984.

[4] G. Samorodnitsky and M. Taqqu, *Stable non Gaussian Processes*, New York: Chapman and Hall, 1994.

[5] K. Panki and S. Renming, "Stable process with singular drift," *Stochastic Process. Appl.*, vol. 124, no. 7, pp. 2479-2516, 2014.

[6] C. Zhen, N. Qing, and W. Longmin, "Uniqueness of stable processes with drift," *Proc. Amer. Math. Soc*., vol. 144, pp. 2661-2675, 2016.

[7] E. Sousa, "Performance of a spread spectrum packet radio network link in a poisson of interferences," *IEEE Trans. Inform. Theory*, vol. 38, pp. 1743-1754, 1992.

[8] M. Shao and C. L. Nikias, "Signal processing with fractional lower order moments: Stable processes and their applications," *Proc. IEEE*, vol. 81, no. 7, pp. 986-1010, 1993.

[9] C. L. Nikias and M. Shao, *Signal Processing with Alpha-Stable Distributions and Applications*, New York: Wiley, 1995.

[10] F. Wen, "Diffusion least-mean p-power algorithms for distributed estimation in alpha-stable noise environments," *Electronics Letters*, vol. 4, pp. 1355-1356, 2013.

[11] N. Azzaoui, L. Clavier, and R. Sabre, "Path delay model based on stable distribution for the 60GHz indoor channel," in *Proc. IEEE GLOBECOM*, 2002, pp. 441-467.

[12] J. P. Montillet and Y. Kegen, "Modeling geodetic processes with levy alpha-stable distribution and FARIMA," *Mathematical Geosciences*, vol. 47, no. 6, pp. 627-646, 2015.

[13] M. Pereyra and H. Batalia, "Modeling ultrasound echoes in skin tissues using symmetric alpha-stable processes," *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, vol. 59, no. 1, pp. 60-72, 2012.

[14] X. Zhong and A. B. Premkumar, "Particle filtering for acoustic source tracking in impulsive noise with alpha-stable process," *IEEE Sensors Journal*, vol. 13, no. 2, pp. 589-600, 2012.

[15] W. Ligang and W. Zidong, "Filtering and control for classes of two-dimensional systems," *The Series Studies in Systems of, Decision and Control*, vol. 18, pp. 1-29, 2015.

[16] F. Brice, F. Pene, and M. Wendler, "Stable limit theorem for u-statistic processes indexed by a random walk," *Electron. Commun. Prob*., vol. 22, no. 9, pp.12-20, 2017.

[17] R. Sabre, "Alpha stable filter and distance for multifocus image fusion," *Int. J. Signal Process. Syst.*, vol. 7, no. 2, pp. 66-72, 2019.

[18] R. Sabre, "Spectral density estimate for stationary symmetric stable random field," *Applicationes Mathematicaes*, vol. 23, no. 2, pp. 107-133, 1995.

[19] R. Sabre, "Spectral density estimate for alpha-stable p-adic processes," *Revisita Statistica*, vol. 72, no. 4, pp. 432-448, 2012.

[20] N. Demesh, "Application of the polynomial kernels to the estimation of the spectra of discrete stable stationary processes," *Akad. Nauk. Ukrain. S.S.R. Inst. Mat*., vol. 64, pp. 12-36, 1988.

[21] R. Sabre, "Aliasing free and additive error in spectra for alpha stable signals," *International Journal of Electrical and Computer Engineering*, vol. 13, no. 10, pp. 668-673, 2019.

[22] S. Arlot and A. Celisse, "A survey of cross-validation procedures for model selection," *Statistics Surveys*, vol. 4, pp. 40-79, 2010.

[23] A. W. Bowman, "An alternative method of cross-validation for smoothing of density estimates," *Biometrika*., vol. 71, no. 2, pp. 253-360,1984.

[24] G. Gregoire., "Least squares cross-validation for counting process intensities," *Scand. J. Statis*., vol. 20, pp. 343-360, 1993.

[25] M. Rachdi, "Choix de la largeur de fenêtre spectral par validation croisée pour un processus stationnaire à temps continue," *C. R. Acad. Sci. Paris,* vol. 327, pp. 777-780, 1998.

[26] S. Watanabe, "Asymptotic equivalence of bayes cross validation and widely applicable information criterion singular," *Learning Theory*, pp. 3571-3594, 2010.

[27] J. Ilow and D. Hatzinakos, "Analytic alpha-stable noise modeling in a poisson field of interferers or scatterers," *IEEE Transactions on Signal Processing*, vol. 46, no. 6, pp. 1601-1611, 1998.

[28] A. Janicki and A. Weron, *Simulation and Chaotic Behavior of Alpha-stable Stochastic Processes, Series: Chapman and Hall/CRC Pure and Applied Mathematics*, New York: Marcel Dekker, 1993.

Rachid Sabre received the PhD degree in statistics from the University of Rouen, France, in 1993 and Habilitation to direct research (HdR) from the University of Burgundy, Dijon, France, in 2003. He joined Agrosup Dijon, France, in 1995, where he is an Associate Professor. From 1998 through 2010, he served as a member of Institut de Mathématiques de Bourgogne, France. He was a member of the Scientific Council AgroSup Dijon from 2009 to 2013. From 2012 to 2019, he has been a member of Laboratoire Electronic, Informatique, and Image (LE2I), France. Since 2019 he has been a member of Laboratory Biogeosciences UMR CNRS University Burgundy. He is author/co-author of numerous papers in scientific and technical journals and conference proceedings. His research interests lie in areas of statistical process and spectral analysis for signal and image processing.