Aliasing-Free and Additive Error in Mixed Spectra for Stable Processes. Application: Sound of a Bird just Captivated in Stress

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***Abstract*—Consider a symmetric continuous time α stable process observed with an additive constant error. The objective of this paper is to give a non-parametric estimator of this error by using discrete observations. As the time of process is continuous and the observations are discrete, we encountered the aliasing phenomenon. Our process sample is taken in a way to circumvent the difficulty related to aliasing and we smoothed the periodogram by using Jackson Kernel. The rate of convergence of this estimator is studied when the spectral density is zero at origin. Few long memory processes are taken here as examples. We have applied our estimator to the concrete case of modeling noise of a bird captured under stress.**

***Index Terms*—spectral density, Jackson kernel, stable processes.**

I. Introduction

This work is interested in the class of symmetric alpha stable signals which are known for being infinite energy signals. These processes have been developed in recent decades by several authors, including [1]-[12] to name just a few.

The Gaussian density distribution remains a particular cases of alpha-stable distribution (α =2).

Alpha stable distribution can be considered as the best model for signals that are normlly impulsive. It is used for modeling many phenomena where the Gaussian is not a reasonable choice(when variance is very large). Signals in this class contain high-pitched bursts or occasional spikes.

Symmetric alpha stable processes are used for modeling many phenomenons in several fields: physics, biology, electronic and electrical engineering, hydrology, economies, communications and radar applications and signal image processing,… see [13]-[24].

In this work, we consider a symmetric α stable harmonizable process $Z = \{Z_t : t \in R\}$. Alternatively Z has the integral representation:

$$Z_t = \int_{-\infty}^{\infty} \exp[i(t\lambda)] d\xi(\lambda)$$

where $1 < \alpha < 2$ and ξ is a complex valued symmetric α-stable random measure on R with independent and isotropic increments. The paper [4] defined the measure $m(A) = |\xi(A)|_\alpha^\alpha$ called "control" measure or spectral measure. Suppose that this measure is absolutely continuous with respect to Lebegue measure: $md(x) = \phi(x)dx$. The function ϕ is called the spectral density. The estimations of the spectral density are given: in [4] when the time of the process is continuous, in [25] when the time of the process is discrete and in [26] when the time of the process is p-adic.

This paper considers a case frenquently encountred in event where we can not directly observe the process Z but we observe such process with an unknown additional constant error. The observed process is $X_t = a + Z_t$ instead of the process Z alone.

The objective of this work is to give an estimator of this constant a, in the specific case when the spectral measure is mixed : the sum of an absolutely continuous measure and a discrete measure:

$$d\mu(\lambda) = \phi(x)dx + \sum_{i=1}^{q} c_i \delta_{w_i}$$

where δ is a Dirac measure, ϕ is nonnegative integrable and bounded function. c_i is unknown positive real number and w_i is unknown real number. Assume that $w_i \neq 0$. [27] gives the estimation of the error "a" when the time of the process is discrete. In this paper, we consider that the time of the process is continuous.We can not use the same estimator given for discrete time because we will enconter aliasing phenomenon. Our objective is to propose a nonparametric estimator of the error "a" after discrete sampling of the process $X(t)$. This work is motived by the fact that it is pratically impossible to observe the process over a continuous time interval. However, we sampled the process at equidistant times, i.e. $t_n = n\tau$, $\tau > 0$. It is

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known that aliasing phenomenon occurs. For more details about aliasing phenomenon, see [28]. We assume that the spectral density ϕ is null for $|\lambda| > \Omega$ where Ω is a nonnegative real number. Moreover this assumption is verified for the alpha stable density. We give an estimate of the addive error and study the convergence rate where the spectral density is zero at origin, particularly at $\phi(\lambda) = \sin^{2k\alpha}\left(\frac{\lambda}{2}\right) g(\lambda)$ and $\phi(\lambda) = |\lambda|^{\beta} g(\lambda)$. We proove that the of convergence rate is improuved following the value of β.

This paper is organized as follows: The second section gives some definitions and proprieties of symmetric stable processes and an estimator of the constant a that we show to be converging in probability to a and to be converging in L_p $(p < \alpha)$ to replace the convergence in mean square because the second moment of the processes is infinite. In third section, we improve the rate of convergence when the spectral density of Z is assumed zero at origin precisely$\phi(\lambda) = |\lambda|^{\beta} g(\lambda)$. The fourth section, is devoted to numerical studies. The last section, deals with the conclusion and perspective of the work.

II. The Estimate of the Constant Error

Starting with the introduction of some basic notations and properties of alpha stable distribution and process. A random variable X is symmetric α-stable (SαS), $0 < \alpha < 2$, if its characteristic function is defined by:

$$\phi_X(\theta) = e^{-\sigma^{\alpha}|\theta|^{\alpha}}$$

where α is the characteristic exponent and σ is the dispersion of the distribution. When α takes the values 1 and 2, we obtain two important special cases of (SαS) distribution, namely Cauchy distribution and Gaussian distribution.

The random variables $X_1, \dots, X_d$ are jointly (SαS) if there is a single positive measure $\Gamma_{X^{(d)}}$ on S_d, unit sphere of R^d, where its characteristic function is of the form:

$$\phi_X(\theta_1, \dots, \theta_d) = \exp\left\{-\int_{S_d} |\theta_1 s_1 + \cdots + \theta_d s_d|^{\alpha} d\Gamma_{X^{(d)}}(s_1, \dots, s_d)\right\}.$$

When X_1 and X_2 are jointly (SαS), the covariation of (X_1, X_2) is defined in [1] by

$$[X_1, X_2]_{\alpha} = \int_{S_2} s_1 . (s_2)^{<\alpha-1>} d\Gamma_{X_1,X_2}(s_1, s_2)$$

where $s^{<\beta>} = sign\,(s).|s|^{\beta}$. Since the moment of second order is infinity, The covariation plays the same role as the covariance.

From the definition of the covariation, [8] defined the following norm on the linear space of (SαS) random variables:

$$|X|_{\alpha} = [X, X]_{\alpha}^{1/\alpha}$$

The process $\xi = (\xi_t, t \in R)$ is symmetric α-stable if all linear combinations $\sum_{i=1}^{n} \lambda_i \xi_{t_i}$ are (SαS) variables.

This paper considers a (SαS) process where its spectral representation is

$$Z_t = \int_{-\infty}^{\infty} e^{it\lambda} d\xi(\lambda),$$

where ξ is a isotropic symmetric α -stable with independent increments. The measure defined by: $\mu(]s,t]) = |\xi(t) - \xi(s)|_{\alpha}^{\alpha}$ is Lebesgue-Stiel measure called the spectral measure (see [1] and [4]). When μ is absolutely continuous $d\mu(x) = f(x)dx$, the function f is called the spectral density of the process Z.

As in [29], [12] and [25], we give the definition of the Jackson polynomial kernel: Let $Z_1, \dots, Z_N$ N observations of the process, where N satisfies:

$$N - 1 = 2k(n-1) \text{ with } n \in N, \qquad k \in N \cup \left\{\frac{1}{2}\right\}$$

if $k = \frac{1}{2}$ then $n = 2n_1 - 1, n_1 \in N$.

The Jackson's polynomial kernel is defined by: $|H_N(\lambda)|^{\alpha} = \left|A_N H^{(N)}(\lambda)\right|^{\alpha}$ where

$$H^{(N)}(\lambda) = \frac{1}{q_{k,n}}\left(\frac{\sin\left(\frac{n\lambda}{2}\right)}{\sin\left(\frac{\lambda}{2}\right)}\right)^{2k} \quad \text{with}$$

$$q_{k,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\sin\left(\frac{n\lambda}{2}\right)}{\sin\left(\frac{\lambda}{2}\right)}\right)^{2k} d\lambda.$$

In addition, we have $A_N = (B_{\alpha,N})^{-1\alpha}$ with $B_{\alpha,N} = \int_{-\pi}^{\pi}\left|H^{(N)}(\lambda)\right|^{\alpha} d\lambda$.

We give the following lemmas proved in [25] [29], which are used in the reminder of this paper.

Lemma 1.

There is a nonnegative function h_k such as:

$$H^{(N)}(\lambda) = \sum_{m=-k(n-1)}^{k(n-1)} h_k\left(\frac{m}{n}\right)\cos(m\lambda)$$

Lemma 2.

Let $B'_{\alpha,N} = \int_{-\pi}^{\pi}\left|\left(\frac{\sin\left(\frac{n\lambda}{2}\right)}{\sin\left(\frac{\lambda}{2}\right)}\right)\right|^{2k\alpha} d\lambda$ *and* $J_{N,\alpha} = \int_{-\pi}^{\pi}|u|^{\gamma}|H_N(\lambda)|^{\alpha}\, d\lambda$, *where* $\gamma \in]0,2]$.

Then

$$B'_{\alpha,N} \geq (2/\pi)^{2k\alpha} n^{2k\alpha-1} \quad if \quad 0 < \alpha < 2$$

$$B'_{\alpha,N} \leq \frac{4\pi k\alpha}{2k\alpha - 1}\, n^{2k\alpha-1} \; if \; \frac{1}{2k} < \alpha < 2$$

and

$$J_{N,\alpha} \leq \frac{\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma - 2k\alpha + 1)}\frac{1}{n^{2k\alpha-1}} \quad if \quad \frac{1}{2k} < \alpha < \frac{\gamma+1}{2k}$$

$$J_{N,\alpha} \leq \frac{2k\alpha\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma+1)(2k\alpha - \gamma - 1)}\frac{1}{n^{\gamma}} \quad if \quad \frac{\gamma+1}{2k} < \alpha < 2$$

In this paper, we propose an estimate of the constant error a defined by:

$$\hat{a} = \frac{\tau A_N}{H_N(0)}\sum_{n'=-k(n-1)}^{k((n-1)} h_k\left(\frac{n'}{n}\right) X\big(\tau n' + \tau k(n-1)\big). \quad (1)$$

Theorem 3.

Let p a real number such that $0 < p < \alpha$. Then

$$|\hat{a} - a|^p = O\left(n^{-\frac{p}{\alpha}}\right)$$

Proof

From the spectral representation of the process, the estimator proposed becomes

$\hat{a} = \frac{\tau A_N}{H_N(0)} \sum_{n'=-k(n-1)}^{k((n-1)} W(n') + a$, where

$W(n') = \left[h_k\left(\frac{n'}{n}\right) \int_{-\infty}^{\infty} exp[i([\tau n' + \tau k(n-1)]\lambda)] d\xi(\lambda)\right]$

Using [1], the characteristic function of $(\hat{a} - a)$ can be written as:

$$E\exp[i\Re e\bar{r}(\hat{a} - a)] = \exp\left[-C_\alpha |r|^\alpha) \left|\int_{-\infty}^{\infty} \frac{\tau^{\frac{1}{\alpha}} A_N}{H_N(0)} \mathrm{W(n')}\right|^\alpha d\xi(\lambda)\right].$$

where $r = r_1 + ir_2$. It is easy to show that:

$$E\exp[i\Re e\bar{r}(\hat{a} - a)] = \exp(-C_\alpha |r|^\alpha \psi_N),$$

where $\psi_N = \psi_{N,1} + \psi_{N,2}$ with

$\psi_{N,1} = \tau \int_{-\infty}^{\infty} \frac{|H_N(\tau\lambda)|^\alpha}{|H_N(0)|^\alpha} \phi(\lambda) d\lambda$ and

$$\psi_{n,2} = \tau \sum_{i=1}^{q} c_i \frac{|H_N(\tau w_i)|^\alpha}{|H_N(0)|^\alpha}$$

Putting $\tau\lambda = x$, we have

$\psi_{N,1} = \int_{-\infty}^{\infty} \frac{|H_N(x)|^\alpha}{|H_N(0)|^\alpha} \phi\left(\frac{x}{\tau}\right) dx$. On the other hand,

$$\psi_{N,1} = \sum_{j \in Z} \int_{(2j-1)\pi}^{2j+1)\pi} \frac{|H_N(x)|^\alpha}{|H_N(0)|^\alpha} \phi\left(\frac{x}{\tau}\right) dx$$

H_N is 2π-periodic, we get

$$\psi_{N,1} = \sum_{j \in Z} \int_{-\pi}^{\pi} \frac{|H_N(x)|^\alpha}{|H_N(0)|^\alpha} \phi_j\left(\frac{x}{\tau}\right) dx$$

where $\phi_j(x) = \phi\left(\frac{x - 2\pi j}{\tau}\right)$. Let j be an integer such that $-\Omega < \frac{x - 2\pi j}{\tau} < \Omega$. Since $\tau\Omega < \pi$ and $|x| < \pi$, we get $|j| < \frac{\tau\Omega}{2\pi} + \frac{1}{2} < 1$ and then

$j = 0$. Therefore

$$\psi_{N,1} = \int_{-\pi}^{\pi} \frac{|H_N(x)|^\alpha}{|H_N(0)|^\alpha} \phi\left(\frac{x}{\tau}\right) dx. \quad (2)$$

The function ϕ being bounded on $[-\pi, \pi]$ and $|H_N(.)|^\alpha$ being a kernel, it can be shown that $\int_{-\pi}^{\pi} |H_N(\lambda)|^\alpha \phi(\lambda) d\lambda$ is converging to $\phi(0)$. On the other hand, from lemma 2, we have:

$$\frac{1}{|H_N(0)|^\alpha} = B'_{\alpha,N} n^{2k\alpha} = O\left(\frac{1}{n}\right)$$

Therefore $\psi_{N,1}$ converges to 0.

$$\psi_{n,2} \leq \sum_{i=1}^{q} \frac{c_i}{B'_{\alpha,N}} \frac{1}{\left|\sin\left[\frac{1}{2}\tau w_i\right]\right|^{2k\alpha}} \frac{B'_{\alpha,N}}{n^{2k\alpha}}.$$

Therefore

$$\psi_{N,2} = O\left(\frac{1}{n^{2k\alpha}}\right).$$

As $2k\lambda > 1$, we obtain

$$\psi_N = O\left(\frac{1}{n}\right). \quad (3)$$

Consequently, the characteristic function of $\hat{a} - a$ converges to 1 when N tends to infinity. Hence $\hat{a}$ converges in probability of to a.

We study now the convergence of $\hat{a}$ to a in L_p where $0 < p < \alpha$, which replaces the convergence in mean square, because the second order moment of X is infinity.

Let

$$D_p = \Re e \int_{-\infty}^{\infty} \int_{-\pi/4}^{\pi/24} \frac{1 - e^{ir\cos\theta}}{|r|^{1+p}} dr d\theta.$$

$$D_p E|\hat{a} - a|^p = \left(\frac{\pi}{2}\right) \int_{-\infty}^{\infty} \frac{1 - e^{-C_\alpha |t|^\alpha \psi_N}}{|t|^{1+p}} dt.$$

Let $u = t[\psi_N]^{\frac{1}{\alpha}}$ and using (3), we obtain

$$\frac{2}{\pi} C_{p,\alpha} E|\hat{a} - a|^p = (\psi_N)^{\frac{p}{\alpha}} = O\left(\frac{1}{n^{\frac{p}{\alpha}}}\right). \quad (4)$$

where

$$C_{p,\alpha} = R_p F_{p,\alpha}^{-1} (C_\alpha)^{-\frac{p}{\alpha}}$$

with

$$R_p = \int \frac{(1 - \cos(u))}{|u|^{1+p}} du$$

and $F_{p,\alpha} = \int \frac{1 - e^{-|u|^\alpha}}{|u|^{\frac{1+p}{\alpha}}} du.$

III. THE IMPROVEMENT OF THE RATE OF CONVERGENCE

In order to improve the convergence rate of the estimator $\hat{a}$, we consider the cases where the spectral density is zero at the origin.

Theorem 4.Assume that the spectral density is satisfying:

$$\phi(\lambda) = |\lambda|^\beta g(\lambda)$$

where $\in]0, 2k\alpha - 1[$, $\lambda \in [-\pi, \pi]$, $\tau w_i \notin 2\pi Z$ and $g(\lambda)$ is a bounded function on $[-\pi, \pi]$, continuous in neighborhood of 0 and $g(0) \neq 0$. Then

$$\frac{2^{4kp}}{\tau^\beta} L \leq \lim_{N \to \infty} n^{\frac{p(\beta+1)}{\alpha}} E|\hat{a} - a|^p \leq \frac{2^{4kp}}{\tau^\beta} L,$$

where L is the following constant:

$$L = \frac{\pi}{2C_{p,\alpha}} \left[g(0) \int_{-\infty}^{\infty} \frac{\left|\sin\frac{u}{2}\right|^{2k\alpha}}{|u|^{2k\alpha-\beta}} du\right]^{p\alpha}.$$

Proof:

From the definition of H_N and (2), the function ψ_N can be written as:

$$\psi_N = n^{-2k\alpha}\int_{-\pi}^{\pi}\left|\frac{\sin\frac{n\lambda}{2}}{\sin\frac{\lambda}{2}}\right|^{2k\alpha}\left|\frac{\lambda}{\tau}\right|^{\beta} g\left(\frac{\lambda}{\tau}\right)d\lambda + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}$$

Using the following inequality:

$$\left|\sin\frac{x}{2}\right| \geq \frac{x}{\pi} \qquad 0 \leq x \leq \pi, \tag{5}$$

we maximize ψ_N as follows:

$$\psi_N \leq \frac{\pi^{4k\alpha}n^{-2k\alpha}}{\tau^{\beta}}\int_{-\pi}^{\pi}\frac{\left|\sin\frac{n\lambda}{2}\right|^{2k\alpha}}{|\lambda|^{2k\alpha-\beta}} g\left(\frac{\lambda}{\tau}\right)d\lambda + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{1}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}.$$

Putting $n\lambda = u$, we have

$$\psi_N \leq \frac{\pi^{4k\alpha}n^{-1-\beta}}{\tau^{\beta}}\left[\int_{-\infty}^{\infty}\frac{\left|\sin\frac{u}{2}\right|^{2k\alpha}}{|u|^{2k\alpha-\beta}} g\left(\frac{u}{\tau n}\right)du + \frac{n^{-2k\alpha+1+\beta}}{\pi^{4k\alpha}}\sum_{i=1}^{q} c_i\left|\frac{1}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}\right].$$

On the other hand, using Lebesgue's dominated convergence theorem, we show that:

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\frac{\left|\sin\frac{u}{2}\right|^{2k\alpha}}{|u|^{2k\alpha-\beta}} g\left(\frac{u}{\tau n}\right)du + \frac{n^{-2k\alpha+1+\beta}}{\pi^{4k\alpha}}\sum_{i=1}^{q} c_i\left|\frac{1}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha} = g(0)\int_{-\infty}^{\infty}\frac{\left|\sin\frac{u}{2}\right|^{2k\alpha}}{|u|^{2k\alpha-\beta}}du. \tag{6}$$

Lemma 2 gives:

$$\lim_{N\to\infty} n^{\frac{p(\beta+1)}{\alpha}}(\psi_N)^{\frac{p}{\alpha}} \leq \frac{\pi^{4kp}}{\tau^{\beta}}\left(g(0)\int_{-\infty}^{\infty}\frac{\left|\sin\frac{u}{2}\right|^{2k\alpha}}{|u|^{2k\alpha-\beta}}du\right)^{\frac{p}{\alpha}}.$$

Thus ψ_N converges to zero. Using the following inequality

$$|\sin x| \leq |x| \qquad \forall x \in [-\pi,\pi], \tag{7}$$

we obtain:

$$\psi_N \geq \frac{2^{4k\alpha}n^{-2k\alpha}}{\tau^{\beta}}\int_{-\pi}^{\pi}\frac{\left|\sin\frac{u}{2}\right|^{2k\alpha}}{|u|^{2k\alpha-\beta}} g\left(\frac{u}{\tau}\right)du + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}.$$

$$\psi_N \geq \frac{2^{4k\alpha}n^{-\beta-1}}{\tau^{\beta}}\left[\int_{-\pi}^{\pi}\frac{\left|\sin\frac{u}{2}\right|^{2k\alpha}}{|u|^{2k\alpha-\beta}} g\left(\frac{u}{\tau n}\right)du + R_n\right],$$

where $R_n = \frac{n^{-2k\alpha+\beta+1}}{2^{4k\alpha}}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}$. Since R_n converges to zero, the equality (5) gives:

$$\lim_{N\to\infty} n^{\frac{p(\beta+1)}{\alpha}}(\psi_N)^{\frac{p}{\alpha}} \geq \frac{2^{4kp}}{\tau^{\beta}}\left(g(0)\int_{-\infty}^{+\infty}\frac{\left|\sin\frac{u}{2}\right|^{2k\alpha}}{|u|^{2k\alpha-\beta}}du\right)^{p\alpha}.$$

The first equality of (4) reaches the result of this theorem.

Theoem 5. Assuming that the spectral density satisfies:

$$\phi(\lambda) = sin^{2k\alpha}\left(\frac{\lambda}{2}\right)g(\lambda)$$

where the function g is integrable on $[-\pi,\pi]$ and $g(0) \neq 0$. Then

$$cte\frac{\pi}{2C_{p,\alpha}}\left(\int_{-\pi}^{\pi} g\left(\frac{\lambda}{\tau}\right)d\lambda\right)^{\frac{p}{\alpha}} \leq \lim_{N\to\infty} n^{2pk}E|\hat{a}-a|^p \leq \frac{\pi}{2C_{p,\alpha}}\left(\int_{-\pi}^{\pi} g(\lambda)d\lambda + \sum_{i=1}^{q} c_i\left|\frac{1}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}\right)^{\frac{p}{\alpha}}.$$

Proof: From the definition of ψ_N and (2), we have

$$\psi_N = n^{-2k\alpha}\int_{-\pi}^{\pi}\left|\sin\frac{n\lambda}{2}\right|^{2k\alpha}\left|\frac{\sin\frac{\lambda}{2\tau}}{\sin\frac{\lambda}{2}}\right|^{2k\alpha} g\left(\frac{\lambda}{\tau n}\right)d\lambda + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}.$$

As $1 \leq k\alpha$ and the sinus function is increasing in $[\pi/2,\pi/2]$, the next expression is bounded by

$$\psi_N \leq n^{-2k\alpha}\int_{-\pi}^{\pi}\left|\sin\frac{\lambda n}{2}\right|^{2} g\left(\frac{\lambda}{n\tau}\right)d\lambda + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{nw_i}{2}\right]}{\sin\left[\frac{w_i}{2}\right]}\right|^{2k\alpha}$$

$$\psi_N \leq n^{-2k\alpha}\left[\int_{-\pi}^{\pi} g(\lambda)d\lambda + \sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}\right].$$

So, from lemma 2, we have:

$$\lim_{N\to\infty}\psi_N n^{2k\alpha} \leq \int_{-\pi}^{\pi} g(\lambda)d\lambda + \sum_{i=1}^{q} c_i\left|\frac{1}{\sin\left[\frac{w_i}{2}\right]}\right|^{2k\alpha}.$$

Using the fact that the sinus function is between -1 and 1 and that $k\alpha < [k\alpha]+1$ where $[k\alpha]$ represents the integer part of $k\alpha$, we obtain

$$\psi_N \geq n^{-2k\alpha}\int_{-\pi}^{\pi}\left[\left(\sin\frac{n\lambda}{2}\right)^2\right]^{[k\alpha]+1} g\left(\frac{\lambda}{\tau}\right)d\lambda + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}$$

$$\psi_N \geq n^{-2k\alpha}\frac{n^{-1}}{2B'_{\alpha,N}}\int_{-\pi}^{\pi}\left[\frac{1-\cos n\lambda}{2}\right]^{[k\alpha]+1} g\left(\frac{\lambda}{\tau}\right)d\lambda + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha}.$$

The binomial formula gives:

$$2^{[k\alpha]+1}\psi_N \geq n^{-2k\alpha}\int_{-\infty}^{\infty} g(\lambda)d\lambda + n^{-2k\alpha}\sum_{r=1}^{[k\alpha]+1}(-1)^r C_{[k\alpha]+1}^{r}\int_{-\pi}^{\pi}\cos^r(n\lambda)g\left(\frac{\lambda}{\tau}\right)d\lambda + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau}{w_i}\,2\right]}\right|^{2k\alpha}$$

We know that $\int_{-\pi}^{\pi}\cos^r(n\lambda)g\left(\frac{\lambda}{\tau}\right)d\lambda$ is converging to a constant. Indeed, using the binomial formula, we obtain:

$$\cos^r(n\lambda) = \left(\frac{e^{in\lambda}+e^{-in\lambda}}{2}\right)^r = \frac{1}{2^r}\sum_{j=0}^{r} C_r^j e^{ijn\lambda}e^{-i(r-j)n\lambda}$$

Hence

$$\int_{-\pi}^{\pi}\cos^r(n\lambda)g\left(\frac{\lambda}{\tau}\right)d\lambda = \frac{1}{2^r}\sum_{j=0}^{r} C_r^j\int_{-\pi}^{\pi}\cos((2j-r)n\lambda)g\left(\frac{\lambda}{\tau}\right)d\lambda.$$

The right side of the last equality converges to $\frac{1}{2^r}\int_{-\pi}^{\pi} g\left(\frac{\lambda}{\tau}\right)d\lambda$ when r is even, and converges to 0 when r is odd. Consequently,

$$\lim_{N\to\infty}\sum_{r=1}^{\frac{[k\alpha]+1}{2}}(-1)^r C_{[k\alpha]+1}^{r}\int_{-\pi}^{\pi}\cos^r(n\lambda)g\left(\frac{\lambda}{\tau}\right)d\lambda = \sum_{p=1}^{\frac{[k\alpha]+1}{2}}(-1)^{2p}C_{[k\alpha]+1}^{2p}\frac{1}{2^{2p}}\int_{-\pi}^{\pi} g(\lambda\tau)d\lambda \quad if \;\; [k\alpha]+1 \;\; is\ even$$

and

$$\lim_{N\to\infty}\sum_{r=1}^{[k\alpha]+1}(-1)^r C_{[k\alpha]+1}^{r}\int_{-\pi}^{\pi}\cos^r(n\lambda)g\left(\frac{\lambda}{\tau}\right)d\lambda = \sum_{p=1}^{[k\alpha]2}(-1)^{2p}C_{[k\alpha]+1}^{2p}\frac{1}{2^{2p}}\int_{-\pi}^{\pi} g\left(\frac{\lambda}{\tau}\right)d\lambda \quad else.$$

Since $\lim_{N\to\infty} n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha} = 0$. The similar arguments are used for showing that

$$\lim_{N\to\infty}(\psi_N)^{\frac{p}{\alpha}}n^{2kp} \geq cte\left(\int_{-\pi}^{\pi} g\left(\frac{\lambda}{\tau}\right)d\lambda\right)^{\frac{p}{\alpha}}. \tag{8}$$

Theorem 6.

Assume that spectral density satisfies:

$$\phi(\lambda) = |\lambda|^{\beta}g(\lambda)$$

where $\beta > 2k\alpha - 1$ and g is measurable positive function bounded on $[-\pi,\pi]$. Then

$$cte \times R \leq \lim_{N\to\infty} n^{2kp}E|\hat{a}-a|^p \leq R$$

where $R = \frac{\pi}{2C_{p,\alpha}}\left(\int_{-\pi}^{\pi}\frac{|\lambda|^{\beta}}{\left|sin\frac{\lambda}{2}\right|^{2k\alpha}}g(\lambda)d\lambda\right)^{\frac{p}{\alpha}}$.

Proof:

Define the function l as follows:

$$l(\lambda) = \pi^{2k\alpha} \qquad if \;\; |\lambda| > \pi$$
$$l(\lambda) = |\lambda|^{2k\alpha}|\sin\lambda 2|^{2k\alpha} \quad if \quad 0 < |\lambda| \leq \pi$$
$$l(\lambda) = 2^{2k\alpha} \qquad if\ \lambda = 0.$$

The function ψ_N can be written as:

$$\psi_N = n^{-2k\alpha}\int_{-\pi}^{\pi}\left|\frac{\sin\frac{n\lambda}{2}}{\sin\frac{\lambda}{2}}\right|^{2k\alpha}\sin^{2k\alpha}\left(\frac{\lambda}{2\tau}\right)h\left(\frac{\lambda}{\tau}\right)d\lambda + n^{-2k\alpha}\sum_{i=1}^{q} c_i\left|\frac{\sin\left[\frac{n\tau w_i}{2}\right]}{\sin\left[\frac{\tau w_i}{2}\right]}\right|^{2k\alpha},$$

where $h(\lambda) = l(\lambda)|\lambda|^{\beta-2k\alpha}g(\lambda)$. We know that the function h is integrable on $[-\pi,\pi]$. Indeed, since the function g is bounded, we get:

$$\int_{-\pi}^{\pi} h(\lambda)d\lambda \leq \sup(g)\int_{-\pi}^{\pi} l(\lambda)|\lambda|^{\beta-2k\alpha}d\lambda.$$

Using the inequality (7), we obtain $l(\lambda) \leq \pi^{2k\alpha}$. Thus

$$\int_{-\pi}^{\pi} h(\lambda)d\lambda \leq 2\pi^{2k\alpha}\sup(g)\int_{0}^{\pi}(\lambda)^{\beta-2k\alpha}d\lambda.$$

Since $\beta > 2k\alpha - 1$, the function h is integrable. From (4) and the theorem 4, the result is obtained.

IV. Numerical Studies

In this work, we apply our estimator to the concrete case of studying the sound of a bird that has just been captivated and under stress. This sound was observed with an additive noise supposed to be constant coming from the flapping of wings. We started by studying the variance of this signal, we observe that this variance tends to grow indefinitely following the increasing size of the sample. We also note that the curve of the characteristic function of the signal grows exponentially as a function of time, which led us to modeling by a stable continuous-time alpha process.

Estimation of the alpha parameter is done by using [30], the estimator gives α=1.68.

Using [31], we estimate the spectral density ϕ.

We modeled our signal by a series representation using the work [32],

$$Z_t = C_\alpha(\int \phi(x)dx)^{1\alpha} \sum_{k=1}^{\infty} \varepsilon_k \Gamma_k^{-1\alpha} e^{itV_k} e^{i\theta_k} \text{ where}$$

• ε_k is a sequence of i.i.d. random variables such as: $P[\varepsilon_k = 0] = P[\varepsilon_k = 1] = 12$,

• Γ_k is a sequence of arrival times of Poisson process,

• V_k is a sequence of i.i.d. random variables independent of ε_k and of Γ_k having the same distribution of control measure$m$, which has probability density ϕ.

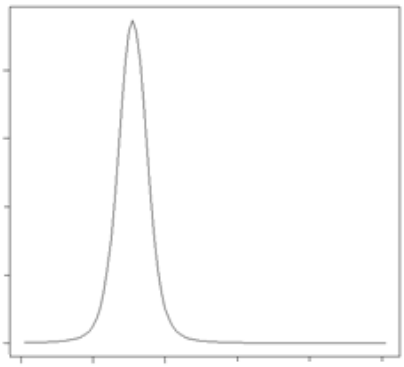

Figure 1. Estimation of the density ϕ.

• θ_k is a sequence of i.i.d. random variables that have the uniform distribution on $[-\pi, \pi]$, independent of ε_k, Γ_k and V_k.

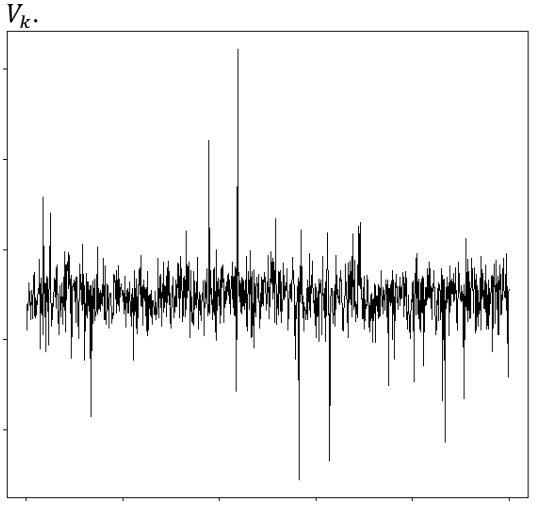

Figure 2. The observed signal.

The observations of the signal is taken at instants: $t_n = \frac{3}{2}n$, then $\tau = \frac{3}{2}$, $0 \leq n \leq N$

Taking $k = 4$, we calculate the estimator $\hat{a}$ given in (1) for different sizes of sample $N = 101, 501, 1001, 1501$, 2001. The result is given in the following table:

N	101	501	1001	1501	2001	2501	3001
$\hat{a}$	14.4	12.5	9.2	10.5	9.7	10.2	10.1

From Table I, the estimator $\hat{a}$ tends to the constant error $a = 10$ when the sample size is large.

V. Conclusion

This work gives a way of solving the problem of the aliasing phenomenon encountered when estimating an error by using discrete observations of continuous-time stable alpha signal. This work could be applied to other processes whose variance is infinite and its observation is disturbed by constant noise. We her give the following examples:

- Segmentation of a sequence of dynamic images, detecting a part of birds in flight;
- detection of possible structural changes in the dynamics of an economic structural phenomenon according to a constant sampling parameter;
- study of the occurrence rate of notes in melodic music in order to simulate the sensation of hearing from afar accompanied by an additional acoustic vibration;
- models of transmission rate in indoor environment for next generation of wireless communication systems.

The perspectives of this work: give a method to find the optimal smoothing parameters such as the cross-validation method.

Conflict of Interest

The authors declare no conflict of interest.

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