Frequency Transformation in Digital Domain

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***Abstract*—This paper introduces a new algorithm to transform a digital low-pass filter into a digital low-pass, high-pass, band-pass, band-stop and narrow-band filter in the digital domain. The technique of this method is based on bilinear and inverse bilinear z-transformation with pre-warping frequency and the support of Pascal's triangle. From this, a matrix equation is derived, called 'Digital low-pass to digital filter Pascal matrix equation'. This equation will make it easier for hand calculations and programming when transformations are made from a digital low-pass filter to another digital filter in the digital domain.**

***Index Terms*—frequency transformation, bilinear z-transformation, digital filters, analog filters, pre-warping frequency, Pascal's triangle**

I. Introduction

One method for designing a digital filter is to transform a given analog low-pass prototype into a desired digital filter by using frequency transformation in the s-domain and bilinear z-transformation [1], [2]. The main advantage of this method is stability from mapping the poles and zeros from the stable region in the s-domain into a stable region in the z-domain. However, when applying bilinear z-transformation, the warping frequency will give a non-linear relation between the analog frequency and the digital frequency [3], [4]. To solve this problem, a method called bilinear z-transformation with pre-warping frequency is used. And also from this method, applying inverse bilinear z-transformation [5], the inverse transformation from a digital filter to an analog low-pass prototype can be found. Thus, it is called inverse bilinear z-transformation with pre-warping frequency. This paper presents a new algorithm for frequency transformation in the digital domain using bilinear and inverse bilinear z-transformation with pre-warping frequency, which can described by a mathematics equation called 'Digital low-pass to digital filter Pascal matrix equation'.

II. Bilinear and Inverse Bilinear z-Transformation with Pre-Warping Frequency

Bilinear z-transformation is a method for transforming an analog filter to an equivalent digital filter [1]. It is defined by (1), where T is a sampling time period, $z=e^{sT}$ and $s=j\omega$. Let ω_A be an angular frequency in the s-domain and ω_D be an angular frequency in the digital domain:

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{1}$$

Equation (1) can be rewritten as:

$$j\omega_A = \frac{2}{T}\frac{1-e^{-j\omega_D T}}{1+e^{-j\omega_D T}} = \frac{2}{T}\tanh\left(j\frac{T}{2}\omega_D\right) = j\tan\left(\frac{T}{2}\omega_D\right)$$
$$\therefore\ \omega_D = \frac{2}{T}\tan^{-1}\left(\frac{T}{2}\omega_A\right) \tag{2}$$

From (2), the relationship between analog frequency ω_A and digital frequency ω_D is a non-linear function caused by the tan function. Pre-warping frequency can be used to overcome this.

Let $s = k\frac{1-z^{-1}}{1+z^{-1}}$ where k is a design parameter and ω_c be a wanted frequency $\omega_c = k\tan\left(\frac{T}{2}\omega_c\right)$, solve for k:

$$\therefore\ k = \frac{\omega_c}{\tan\left(\frac{T}{2}\omega_c\right)}$$

The bilinear z-transformation with pre-warping frequency is:

$$s = \omega_c \cot\left(\frac{T}{2}\omega_c\right)\frac{1-z^{-1}}{1+z^{-1}} \tag{3}$$

A. Frequency Transformation from the s-Domain to the z-Domain with Bilinear z-Transformation with Pre-Warping Frequency

The transfer function H(z) of a digital filter can be obtained from the transfer function H(s) of an analog low-pass prototype filter by first transforming an analog low-pass prototype to an analog filter that is the same class of the digital filter using frequency transform in the s-domain [6], and then applying the bilinear z-transformation with pre-warping frequency [7]-[9]. Table I illustrates the frequency transformation from the s-domain to the z-domain.

Let f_s is the sampling frequency, f_c is the cut-off frequency for the low-pass and high-pass filter, and f_U and f_L are the upper and lower frequency for the band-pass, band-stop and narrow-band filter. The parameters of c, t, U and L can be found as below.

$$c = \cot\left(\pi\frac{f_c}{f_s}\right) \qquad t = \tan\left(\pi\frac{f_c}{f_s}\right)$$

$$\begin{cases} c_U = \cot\left(\pi\frac{f_U}{f_s}\right) \\ t_L = \tan\left(\pi\frac{f_L}{f_s}\right) \end{cases} \Rightarrow \begin{cases} U = \frac{c_U}{1-c_U t_L} \\ L = \frac{t_L}{1-c_U t_L} \end{cases}$$

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In the case of a narrow band, if given the center frequency f_0 and quality factor Q, the upper and lower frequency can be calculated as below.

$$\begin{cases} f_L = f_0\left(\sqrt{1+\dfrac{1}{4Q^2}}-\dfrac{1}{2Q}\right) \\ f_U = f_0\left(\sqrt{1+\dfrac{1}{4Q^2}}+\dfrac{1}{2Q}\right) \end{cases}$$

B. *Inverse Bilinear z-Transformation with Pre-Warping Frequency*

Inverse bilinear z-transformation with pre-warping frequency is a method used to transform a digital filter back to an analog low-pass prototype [10]. In this paper, it shows the transformation from a digital low-pass to an analog low-pass prototype.

TABLE I. FREQUENCY TRANSFORMATION FORM S-DOMAIN TO Z-DOMAIN USING BILINEAR Z-TRANSFORMATION WITH PRE-WARPING FREQUENCY

Converting types	Frequency transformation in s-domain	Bilinear z-transformation with pre-warping frequency
Low pass to low pass	$\dfrac{s}{\omega_c}$	$c\dfrac{1-z^{-1}}{1+z^{-1}}$
Low pass to high pass	$\dfrac{\omega_c}{s}$	$t\dfrac{1+z^{-1}}{1-z^{-1}}$
Low pass to band pass	$\dfrac{s^2+\omega_U\omega_L}{(\omega_U-\omega_L)s}$	$U\dfrac{1-z^{-1}}{1+z^{-1}}+L\dfrac{1+z^{-1}}{1-z^{-1}}$
Low pass to band stop	$\dfrac{(\omega_U-\omega_L)s}{s^2+\omega_U\omega_L}$	$\dfrac{1}{U\dfrac{1-z^{-1}}{1+z^{-1}}+L\dfrac{1+z^{-1}}{1-z^{-1}}}$

In Table I, to transform an analog low-pass prototype with transfer function H(s) to a digital low-pass with transfer function H(z), s in H(s) is replaced by $c\frac{1-z^{-1}}{1+z^{-1}}$. The inverse bilinear z-transformation with pre-warping frequency is then:

$$s = c\frac{1-z^{-1}}{1+z^{-1}} \quad \Rightarrow \quad z^{-1} = \frac{c-s}{c+s} \tag{4}$$

Equation (4) is an inverse bilinear z-transformation with warping frequency for transforming a digital low-pass to an analog low-pass prototype.

III. FREQUENCY TRANSFORMATION IN THE DIGITAL DOMAIN

This section introduces a new algorithm for frequency transformation in the digital domain. From Table I and an inverse bilinear z-transformation with pre-warping frequency, a digital low-pass is transformed into a digital low-pass, high-pass, band-pass, band-stop and narrow-band, as shown in Table II.

In Table II, to transform a digital low-pass with the transfer function H(z) into a digital filter, z is substituted by $Ƶ(z^{-1})$. Then, with the help of Pascal's triangle, a matrix equation is derived, called 'Digital low-pass to digital filter Pascal matrix equation'.

TABLE II. FREQUENCY TRANSFORMATION IN DIGITAL DOMAIN

Converting types	$Ƶ(z^{-1})$
Low pass to low pass	$\dfrac{c-c_N+(c+c_N)z^{-1}}{c+c_N+(c-c_N)z^{-1}}$
Low pass to high pass	$\dfrac{c-t-(c+t)z^{-1}}{c+t-(c-t)z^{-1}}$
Low pass to band pass	$\dfrac{c-U-L+2(U-L)z^{-1}-(U+L+c)z^{-2}}{c+U+L-2(U-L)z^{-1}+(U+L-c)z^{-2}}$
Low pass to band stop	$\dfrac{cU+cL-1-2c(U-L)z^{-1}+(cU+cL+1)z^{-2}}{cU+cL+1-2c(U-L)z^{-1}+(cU+cL-1)z^{-2}}$

IV. DIGITAL LOW-PASS TO DIGITAL FILTER PASCAL MATRIX EQUATION

The digital low-pass to digital filter Pascal matrix equation is used to transform the coefficients a_{lp} and b_{lp} of a digital low-pass filter to the coefficients a_d and b_d of a desired digital filter. This matrix equation is multiplied by some matrices', which are found from the coefficient and binomial expansion in Pascal's triangle, and they are introduced in the next section

A. *Diagonal Matrix D_x*

Matrix D_x is a diagonal matrix with a size of (n+1; n+1), where n is the given n^{th}-ordered digital low-pass filter, as shown below:

$$D_x = \begin{bmatrix} x^n & 0 & 0 & \cdots & 0 \\ 0 & x^{n-1} & 0 & 0 & 0 \\ 0 & 0 & x^{n-2} & 0 & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

B. *Matrix P*

Matrix P has a size of (n+1; n+1) and contains the positive and negative coefficients of Pascal's triangle. There are three different matrix P. Depending on the conversion type, matrix P may be P_{LP}, P_{HBS} or P_{BS}, as shown below:

$$P_{LP} = \begin{bmatrix} \begin{cases} (P_{LP})_{i=1;j=1\to n+1} = \binom{n}{j-1} \\ (P_{LP})_{i=n+1;j=1\to n+1} = (-1)^{j-1}\binom{n}{j-1} \\ (P_{LP})_{i=1\to n+1;j=1} = 1 \\ (P_{LP})_{i=1\to n+1;j=n+1} = (-1)^{i-1} \\ (P_{LP})_{i;j} = -(P_{LP})_{i;j-1} - (P_{LP})_{i-1;j-1} + (P_{LP})_{i-1;j} \end{cases} \end{bmatrix}$$

$$P_{HBS} = \begin{bmatrix} \begin{cases} (P_{LP})_{i=1;j=1\to n+1} = (-1)^{j-1}\binom{n}{j-1} \\ (P_{LP})_{i=n+1;j=1\to n+1} = \binom{n}{j-1} \\ (P_{LP})_{i=1\to n+1;j=1} = 1 \\ (P_{LP})_{i=1\to n+1;j=n+1} = (-1)^{i-1} \\ (P_{LP})_{i;j} = (P_{LP})_{i;j-1} + (P_{LP})_{i-1;j-1} + (P_{LP})_{i-1;j} \end{cases} \end{bmatrix}$$

$$P_{BS}=\left[\begin{cases}\left(P_{LP}\right)_{i=1;j=1\to n+1}=\binom{n}{j-1}\\ \left(P_{LP}\right)_{i=n+1;j=1\to n+1}=(-1)^{j-1}\binom{n}{j-1}\\ \left(P_{LP}\right)_{i=1\to n+1;j=1}=(-1)^{i-1}\\ \left(P_{LP}\right)_{i=1\to n+1;j=n+1}=1\\ \left(P_{LP}\right)_{i;j}=-\left(P_{LP}\right)_{i;j-1}+\left(P_{LP}\right)_{i-1;j-1}-\left(P_{LP}\right)_{i-1;j}\end{cases}\right]$$

C. Matrix T

Two kinds of matrix T—T_{UL} and T_x—are introduced. One useful application of Pascal's triangle is the expansion of binomial $(U+L)^n$. Inserting zeroes will make matrix T_{UL}, with a size of (n+1; 2n+1), and matrix T_x, with a size of (n+1+; n+1) as shown below:

$$T_{UL}=\begin{bmatrix}0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&U&0&L&0&0&0&0\\0&0&0&U^2&0&2UL&0&L^2&0&0&0\\0&0&U^3&0&3U^2L&0&3UL^2&0&L^3&0&0\\\vdots&\iddots&0&\cdots&0&\cdots&0&\cdots&0&\ddots&\vdots\\U^n&0&\cdots&0&\cdots&0&\cdots&0&\cdots&0&L^n\end{bmatrix}$$

$$T_x=\begin{bmatrix}1&0&0&0&0\\0&x^2&0&0&0\\0&0&x^3&0&0\\\vdots&0&0&\ddots&\vdots\\0&0&0&0&x^n\end{bmatrix}$$

The transformation from a digital low-pass to another digital filter is called the digital low-pass to digital filter Pascal matrix equation, as written below:

$$\begin{cases}a_d=[a_{LP}][P_{LP}][D_x][T][P]\\ b_d=[b_{LP}][P_{LP}][D_x][T][P]\end{cases}\tag{5}$$

V. Examples for Design of Digital Filter

This section demonstrates five examples for transforming a digital low-pass to a digital filter using the digital low-pass to digital filter Pascal matrix equation with MATLAB programming as shown in Fig. 1 (low pass to low pass), Fig. 2 (low pass to high pass), Fig. 3 (low pass to band pass), Fig. 4 (low pass to band stop) and Fig. 5 (low pass to Notch).

A. Example 1: Transforming a Digital Low-Pass to a Digital Low-Pass

Transforming a second-ordered Butterworth digital low-pass with the transfer function H(z) at corner frequency 200Hz to a digital low-pass with corner frequency 400Hz at sampling frequency 2000Hz:

$$H(z)=\frac{1+2z^{-1}+z^{-2}}{14.8246-16.9443z^{-1}+6.1196z^{-2}}$$

$$f_c=200Hz\ ,\ f_{cn}=400Hz\ ,\ fs=2000Hz$$

$$c=\cot\left(\pi\frac{f_c}{fs}\right)=3.0777\ ,\ cn=\cot\left(\pi\frac{f_{cn}}{f_s}\right)=1.3764$$

***Apply* (5)**

$$\begin{cases}a_d=a_{LP}P_{LP}D_cT_{cn}P_{LP}\\ b_d=b_{LP}P_{LP}D_cT_{cn}P_{LP}\end{cases}$$

$$a_d=\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}1&2&1\\1&0&-1\\1&-2&1\end{bmatrix}\begin{bmatrix}9.4721&0&0\\0&3.0777&0\\0&0&1\end{bmatrix}$$

$$\times\begin{bmatrix}1&0&0\\0&1.3764&0\\0&0&1.8944\end{bmatrix}\begin{bmatrix}1&2&1\\1&0&-1\\1&-2&1\end{bmatrix}$$

$$=\begin{bmatrix}37.8885&75.7771&37.8885\end{bmatrix}=37.8885\times\begin{bmatrix}1&2&1\end{bmatrix}$$

$$b_d=\begin{bmatrix}14.8246&-16.9443&6.1196\end{bmatrix}\begin{bmatrix}1&2&1\\1&0&-1\\1&-2&1\end{bmatrix}\begin{bmatrix}9.4721&0&0\\0&3.0777&0\\0&0&1\end{bmatrix}$$

$$\times\begin{bmatrix}1&0&0\\0&1.3764&0\\0&0&1.8944\end{bmatrix}\begin{bmatrix}1&2&1\\1&0&-1\\1&-2&1\end{bmatrix}$$

$$=\begin{bmatrix}183.4156&-67.77771&35.9157\end{bmatrix}$$

$$=37.8885\times\begin{bmatrix}4.8409&-1.7889&0.9479\end{bmatrix}$$

$$\therefore H_{LP-new}=\frac{1+2z^{-1}+z^{-2}}{4.8409-1.7889z^{-1}+0.9479z^{-2}}$$

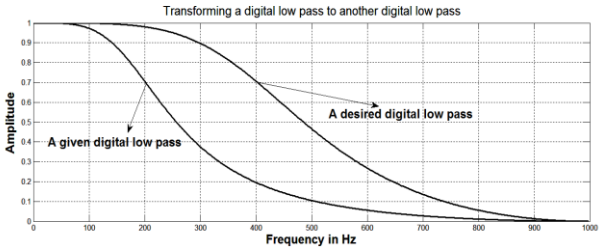

Figure 1. Transforming a digital low-pass to a digital low-pass using the digital low-pass to digital filter Pascal matrix equation.

B. Example 2: Transforming a Digital Low-Pass to a Digital High-Pass

Transforming a second-ordered Butterworth digital low-pass with transfer function H(z) at corner frequency 200Hz to a digital high-pass with corner frequency 200Hz at sampling frequency 2000Hz:

***Apply* (5)**

$$\begin{cases}a_d=a_{LP}P_{LP}D_cT_tP_{HBS}\\ b_d=b_{LP}P_{LP}D_cT_tP_{HBS}\end{cases}$$

$$f_{cn}=200Hz\ ,\ f_s=2000Hz\ \Rightarrow t=\tan\left(\pi\frac{f_{cn}}{f_s}\right)=0.3249$$

$$a_d=\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}1&2&1\\1&0&-1\\1&-2&1\end{bmatrix}\begin{bmatrix}9.4721&0&0\\0&3.0777&0\\0&0&1\end{bmatrix}$$

$$\times\begin{bmatrix}1&0&0\\0&0.3249&0\\0&0&0.1056\end{bmatrix}\begin{bmatrix}1&-2&1\\1&0&-1\\1&2&1\end{bmatrix}$$

$$=\begin{bmatrix}37.8885&-75.7771&37.8885\end{bmatrix}=37.8885\times\begin{bmatrix}1&-2&1\end{bmatrix}$$

$$b_d=\begin{bmatrix}14.8246&-16.9443&6.1196\end{bmatrix}\begin{bmatrix}1&2&1\\1&0&-1\\1&-2&1\end{bmatrix}\begin{bmatrix}9.4721&0&0\\0&3.0777&0\\0&0&1\end{bmatrix}$$

$$\times\begin{bmatrix}1&0&0\\0&0.3249&0\\0&0&0.1056\end{bmatrix}\begin{bmatrix}1&-2&1\\1&0&-1\\1&2&1\end{bmatrix}$$

$$=\begin{bmatrix}59.2986&-67.7771&24.4785\end{bmatrix}$$

$$=37.8885\times\begin{bmatrix}1.5651&-1.7889&0.6461\end{bmatrix}$$

$$\therefore H_{LP-new}=\frac{1-2z^{-1}+z^{-2}}{1.5651-1.7889z^{-1}+0.6461z^{-2}}$$

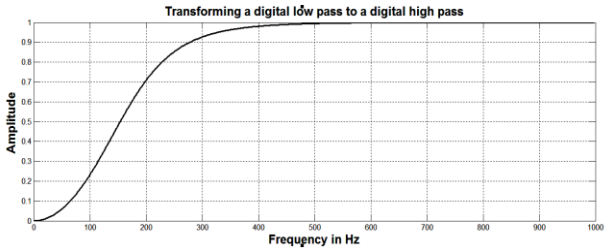

Figure 2. Transforming a digital low-pass to a digital high-pass using the digital low-pass to digital filter Pascal matrix equation.

C. Example 3: Transforming a Digital Low-Pass to a Digital Band-Pass

Transforming a second-ordered Butterworth digital low-pass with transfer function H(z) at corner frequency 200Hz to a digital band-pass with upper frequency 400Hz and lower frequency 200Hz at sampling frequency 2000Hz:

***Apply* (5)**

$$\begin{cases} a_d = a_{LP} P_{LP} D_c T_{UL} P_{HBS} \\ b_d = b_{LP} P_{LP} D_c T_{UL} P_{HBS} \end{cases}$$

$$f_u = 400Hz,\ f_l = 200Hz,\ f_s = 2000Hz$$

$$\begin{cases} c_u = \cot\left(\pi \dfrac{f_u}{f_s}\right) = 1.3764 \\ t_l = \tan\left(\pi \dfrac{f_l}{f_s}\right) = 0.3249 \end{cases} \Rightarrow \begin{cases} U = \dfrac{c_u}{1 - c_u t_l} = 2.4899 \\ L = \dfrac{t_l}{1 - c_u t_l} = 0.5878 \end{cases}$$

$$a_d = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 9.4721 & 0 & 0 \\ 0 & 3.0777 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2.4899 & 0 & 0.5878 & 0 \\ 6.1996 & 0 & 2.9271 & 0 & 0.3455 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$= 37.8885 \times \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$b_d = \begin{bmatrix} 14.8246 & -16.9443 & 6.1196 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 9.4721 & 0 & 0 \\ 0 & 3.0777 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2.4899 & 0 & 0.5878 & 0 \\ 6.1996 & 0 & 2.9271 & 0 & 0.3455 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$= 37.8885 \times \begin{bmatrix} 14.8247 & -28.7964 & 31.4164 & -18.0364 & 6.1196 \end{bmatrix}$$

$$H_{BP}(z) = \frac{1 - 2z^{-2} + z^{-4}}{14.8247 - 28.7984z^{-1} + 31.4164z^{-2} - 18.0364z^{-3} + 6.1196z^{-4}}$$

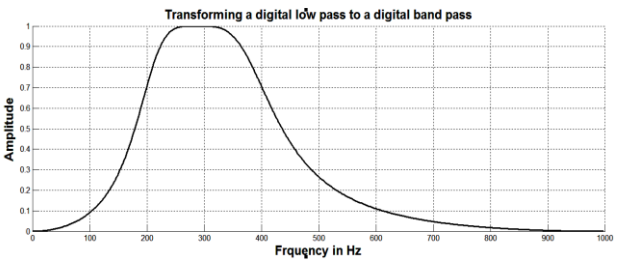

Figure 3. Transforming a digital low-pass to a digital band-pass using the digital low-pass to digital filter Pascal matrix equation.

D. Example 4: Transforming a Digital Low-Pass to a Digital Band-Stop

Transforming a second-ordered Butterworth digital low-pass with transfer function H(z) at corner frequency 200Hz to a digital band-stop with upper frequency 400Hz and lower frequency 200Hz at sampling frequency 2000Hz:

***Apply* (5)**

$$\begin{cases} a_d = a_{LP} P_{BS} D_{c\_inv} T_{UL} P_{HBS} \\ b_d = b_{LP} P_{BS} D_{c\_inv} T_{UL} P_{HBS} \end{cases}$$

$$f_u = 400Hz,\ f_l = 200Hz,\ f_s = 2000Hz$$

$$\begin{cases} c_u = \cot\left(\pi \dfrac{f_u}{f_s}\right) = 1.3764 \\ t_l = \tan\left(\pi \dfrac{f_l}{f_s}\right) = 0.3249 \end{cases} \Rightarrow \begin{cases} U = \dfrac{c_u}{1 - c_u t_l} = 2.4899 \\ L = \dfrac{t_l}{1 - c_u t_l} = 0.5878 \end{cases}$$

$$a_d = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.0777 & 0 \\ 0 & 0 & 9.4721 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2.4899 & 0 & 0.5878 & 0 \\ 6.1996 & 0 & 2.9271 & 0 & 0.3455 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$= 358.9 \times \begin{bmatrix} 1 & -2.472 & 3.5277 & -2.472 & 1 \end{bmatrix}$$

$$b_d = \begin{bmatrix} 14.8246 & -16.9443 & 6.1196 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.0777 & 0 \\ 0 & 0 & 9.4721 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2.4899 & 0 & 0.5878 & 0 \\ 6.1996 & 0 & 2.9271 & 0 & 0.3455 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$= 358.9 \times \begin{bmatrix} 1.565 & -3.04 & 3.3166 & -1.9041 & 0.646 \end{bmatrix}$$

$$H_{BP}(z) = \frac{1 - 2.472z^{-1} + 3.5277z^{-2} - 2.472z^{-3} + z^{-4}}{1.565 - 3.04z^{-1} + 3.3166z^{-2} - 1.9041z^{-3} + 0.646z^{-4}}$$

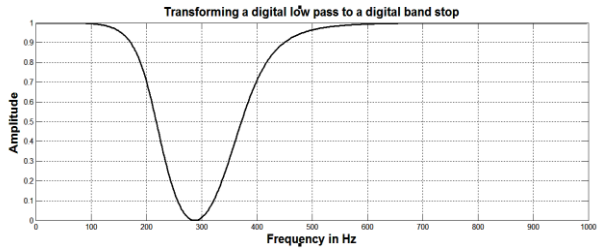

Figure 4. Transforming a digital low-pass to a digital band-stop using the digital low-pass to digital filter Pascal matrix equation

E. Example 5: Transforming a Digital Low-Pass to a Digital Notch Filter

Transforming a second-ordered Butterworth digital low-pass with transfer function H(z) at corner frequency 200Hz to a digital Notch filter with center frequency f_0 = 500Hz and quality factor Q=50 at sampling frequency 2000Hz.

***Apply* (5)**

$$\begin{cases} a_d = a_{LP} P_{BS} D_{c\_inv} T_{UL} P_{HBS} \\ b_d = b_{LP} P_{BS} D_{c\_inv} T_{UL} P_{HBS} \end{cases}$$

$$f_0 = 500Hz,\ Q = 50,\ f_s = 2000Hz$$

$$\begin{cases} f_u = f_0\left(\sqrt{1 + \dfrac{1}{4Q^2}} + \dfrac{1}{2Q}\right) = 505Hz \\ f_l = f_0\left(\sqrt{1 + \dfrac{1}{4Q^2}} - \dfrac{1}{2Q}\right) = 495Hz \end{cases}$$

$$\begin{cases} c_u = \cot\left(\pi \dfrac{f_u}{f_s}\right) = 0.9843 \\ t_l = \tan\left(\pi \dfrac{f_l}{f_s}\right) = 0.9845 \end{cases} \Rightarrow \begin{cases} U = \dfrac{c_u}{1 - c_u t_l} = 31.8259 \\ L = \dfrac{t_l}{1 - c_u t_l} = 31.8309 \end{cases}$$

$$a_d = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.0777 & 0 \\ 0 & 0 & 9.4721 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 31.8259 & 0 & 31.8309 & 0 \\ 1012.9 & 0 & 2026.1 & 0 & 1013.2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$= 10^5 \times \begin{bmatrix} 1.5353 & 0.0005 & 3.0706 & 0.0005 & 1.5353 \end{bmatrix}$$

$$b_d = \begin{bmatrix} 14.8246 & -16.9443 & 6.1196 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.0777 & 0 \\ 0 & 0 & 9.4721 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 31.8259 & 0 & 31.8309 & 0 \\ 1012.9 & 0 & 2026.1 & 0 & 1013.2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$= 10^5 \times \begin{bmatrix} 1.5698 & 0.0005 & 3.0699 & 0.0005 & 1.5016 \end{bmatrix}$$

$$H_{BP}(z) = \frac{1.5353 + 0.0005z^{-1} + 3.0706z^{-2} + 0.0005z^{-3} + 1.5353z^{-4}}{1.5698 + 0.0005z^{-1} + 3.0699z^{-2} + 0.0005z^{-3} + 1.5016z^{-4}}$$

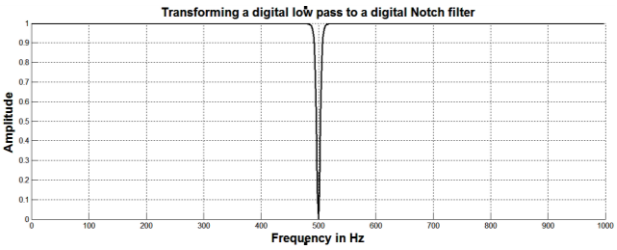

Figure 5. Transforming a digital low-pass to a digital Notch filter using the digital low-pass to digital filter Pascal matrix equation.

VI. Conclusion

A new method was developed for transforming a digital low-pass filter with transfer function H(z) to a digital filter (low-pass, high-pass, band-pass, band-stop and narrow-band). The involvement of Pascal's triangle in the digital low-pass to digital filter Pascal matrix equation, as presented and demonstrated in the examples, made the work easier for hand calculations and computing, when transforming the digital filter in the z-domain. The algorithm of this method is simple because all operations imply the matrix multiplication; therefore, it is more effective in programs and calculations.

Acknowledgment

This is my own research, and I would like to share my idea of how to apply Pascal's triangle in filters design. I would appreciate feedback from readers to improve this research.

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