Realizations of Systems with Point Delays

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***Abstract*—Results on realization theory of dynamic linear systems with lags are obtained through Laurent series expansions. The results are related to controllability and observability properties as well as to mismatching among a real transfer matrix and its nominal version For this purpose, an infinite polynomial block Hankel matrix and associate τ -finite polynomial block matrices are defined in order to relate the spectral controllability and observability properties of minimal realizations with the minimum feasible finite rank of such a Hankel matrix.**

***Index Terms*—realization theory, Hankel matrices, delays**

I. INTRODUCTION

The minimal realization problem of dynamic linear time-invariant delay-free systems is to find a linear state-space description of the minimal possible dimension whose associate transfer matrix exactly matches a proper predefined rational matrix in $\mathbf{K}^{p\times m}(s)$ over a field $\mathbf{K}$. Any proper(i.e. realizable) rational transfer matrix $G(s)\in\mathbf{K}^{p\times m}(s)$ can be expanded in a formal Laurent series at ∞ resulting in the formal identity $G(s)=\sum_{i=0}^{\infty}H_i s^{-i}$, $\{H_i\}_{i\in\mathbf{N}_0}$ (often denoted as $\{H_i\}_0^{\infty}$) being an infinite sequence of matrices which are the block matrices of the infinite block Hankel matrix, with $\mathbf{N}_0=\mathbf{N}\cup\{0\}$ and $\mathbf{N}$ being the set of the natural numbers. The minimal realization problem in the delay-free case may be focused on as finding a state-space realization (A,B,C,D), with $A\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times m}$, $C\in\mathbf{R}^{p\times n}$ and $D\in\mathbf{R}^{p\times m}$, of order $n\in\mathbf{N}$ being minimal (i.e. as small as possible) so that given the infinite sequence $\{H_i\in\mathbf{K}^{p\times m}\}_0^{\infty}$, verifies the identity $G(s)=\sum_{i=0}^{\infty}H_i s^{-i}=C(sI-A)^{-1}B+D$ so that $H_0=D$ and $H_i=CA^{i-1}B$, $i\in\mathbf{N}$. The classical related problem was firstly formulated by Kalman, [1], [2], for the single-input single-output case. The minimal partial realization problem of any approximation τ is formulated as follows: Given a finite given sequence $\{H_i\in\mathbf{K}^{p\times m}\}_0^{\tau}$, some $\tau\in\mathbf{N}$, satisfying $H_0=D$, $H_i=CA^{i-1}B$, $\forall i\in\bar{\tau}:=\{1,2,....,\tau\}$, it exists a quadruple (A,B,C,D), with $A\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times m}$, $C\in\mathbf{R}^{p\times n}$ and $D\in\mathbf{R}^{p\times m}$, of minimal order $n\in\mathbf{N}$ (the order of A) such that $C(sI-A)^{-1}B+D=\sum_{i=0}^{\tau}H_i s^{-i}+0(s^{-\tau-1})$, [3]-[7]. This formalism is extended to the presence of delays.

II. DESCRIPTIONS OF THE SYSTEMS

Consider the linear time-invariant dynamic system in state-space form:

$$\dot{x}(t)=\sum_{i=0}^{q}A_i x(t-h_i)+\sum_{i=0}^{q'}B_i u(t-h_i') \quad (1)$$

$$y(t)=Cx(t)+Du(t) \quad (2)$$

where $x:\mathbf{R}_+\times\mathbf{R}^n\to\Sigma\subset\mathbf{R}^n$, $u:\mathbf{R}_+\times\mathbf{R}^m\to Y\subset\mathbf{R}^m$ and $y:\mathbf{R}_+\times\mathbf{R}^p\to Y\subset\mathbf{R}^p$ are the state, input and output vector functions in their respective state, input and output spaces Σ, U and Y, $\mathbf{R}_+:=\{z\in\mathbf{R}:z\geq 0\}$, $h_0=h_0'=0$ and $\boldsymbol{h}_i\in\mathbf{R}_+$ $(i\in\bar{q}:=\{1,2,...,q\})$, $h_i'\in\mathbf{R}_+$ $(i\in\bar{q}')$ are the internal and external point, in general, incommensurate delays; i.e. h_i and h_j are not necessarily equal to ih_b and $jh_b(i\in\bar{q}, j\in\bar{q}')$, some h_b>0, h_b>0, $A_i\in\mathbf{R}^{n\times n}$ $(i\in\bar{q}\cup\{0\})$, $B_i\in\mathbf{R}^{n\times m}$ $(i\in\bar{q}'\cup\{0\})$, $C\in\mathbf{R}^{p\times n}$ and $D\in\mathbf{R}^{p\times m}$ are matrices of real entries which parameterize the system. The dynamic system is subject to initial conditions $\varphi:[-h,0]\to\mathbf{R}^n$, where $h:=\underset{i\in\bar{q}}{\mathrm{Max}}(h_i)$ being piecewise continuous possibly with bounded discontinuities on a subset of zero measure of its definition domain. By taking right Laplace transforms in the state-space description with $\varphi\equiv 0$, a transfer matrix exists defined by:

$$G(s):=\frac{Y(s)}{U(s)}=\left[\frac{\mathrm{Lap}_+(y(t))}{\mathrm{Lap}_+(u(t))}\right]_{\varphi\equiv 0}$$

$$=C\left(sI_n-\sum_{i=0}^{q}A_i e^{-h_i s}\right)^{-1}\left(\sum_{j=0}^{q'}B_i e^{-h_i' s}\right)+D \quad (3)$$

where $V(s):=\mathrm{Lap}_+(v(t))$ is the right Laplace transform of $v:\mathbf{R}\to\mathbf{R}^s$ provided it exists. Note that G(s)

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is a complex matrix function in $\mathbf{C}^{p\times m}$ in the complex indeterminate s whose (i, j)- th entry is:

$$G_{ij}(s)=\frac{c_i^T \operatorname{Adj}\left(sI_n-\sum_{k=0}^{q} A_i e^{-h_k s}\right)\left(\sum_{j=0}^{q'} b_{\ell j} e^{-h'_\ell s}\right)}{\operatorname{Det}\left(sI_n-\sum_{k=0}^{q} A_i e^{-h_k s}\right)}+D_{ij} \tag{4}$$

where Adj (.) and Det(.) stand for the adjoint matrix and determinant of the (.)-matrix, respectively, and c_i^T and $b_{\ell j}$ being the i-th row of C $(i\in\overline{p})$ and j-th column of B_ℓ $(\ell\in\overline{q}'\cup\{0\})$, respectively. Define complex $(q+q')$ and $(\hat{q}+\hat{q}')$ tuples:

$$z:=\left(z^I, z^E\right)=\left(z_1,\ldots,z_q,z_{q+1},z_{q+q'}\right)\in\mathbf{C}^{q+q'}$$
$$\hat{z}:=\left(\hat{z}^I,\hat{z}^E\right)=\left(z_1,\ldots,z_{\hat{q}},z_{\hat{q}+1},z_{\hat{q}+\hat{q}'}\right)\in\mathbf{C}^{\hat{q}+\hat{q}'} \tag{5}$$

where

$$q\le\hat{q}\le\hat{q}_0:=n\left(\sum_{i=1}^{q}\binom{q}{i}\right);\ q'\le\hat{q}'\le\hat{q}'_0:=\hat{q}_0\left(q'+1\right)$$

Then, $\mathbf{R}^{p\times m}(s,z)$, the space of realizable (i.e. proper) rational transfer p×m matrices of real coefficients in the complex $(\hat{q}+\hat{q}'+1)$- tuple $(s,\hat{z})$ (of numerator and denominator being, respectively, a quasi-polynomial matrix and a quasi-polynomial), is isomorphic (in the sequel denoted with the symbol "$\approx$") to $\mathbf{R}^{p\times m}(s)$ so that there is a natural bijection between each entry $G_{ij}(s)$ and

$$G_{ij}(s,\hat{z})=\frac{N_{ij}(s,\hat{z})}{M(s,\hat{z})}=\frac{\sum_{k=0}^{n_{ij}}\sum_{\ell=0}^{\hat{q}+\hat{q}'} N_{ijk\ell}\, s^k \hat{z}_\ell}{\sum_{k=0}^{n}\sum_{\ell=0}^{\hat{q}} M_{k\ell}\, s^k\hat{z}_\ell} \tag{6}$$

III. Minimal Realizations and Formal Series Descriptions

Note that the numerator and denominator of $G_{ij}(s,\hat{z})$ are, respectively, in the polynomial rings $\mathbf{R}^{p\times m}[s,\hat{z}]$ and $\mathbf{R}[s,\hat{z}]$ generated by $(s,\hat{z})$. By using a formal Laurent series expansion at ∞ in the variable s of the form $G(s,\hat{z})=\sum_{i=0}^{\infty}H_i(\hat{z})s^{-i}$ with $H_i(\hat{z})\in\mathbf{R}^{p\times m}[\hat{z}]$, it follows that $\mathbf{R}^{p\times m}(s,\hat{z})\approx\mathbf{R}^{p\times m}[[s]][\hat{z}]$ (the ring of formal Laurent power series with matrices over $\mathbf{R}^{p\times m}$ at ∞ in the polynomial multiple indeterminate defined by the components of the $\hat{z}$ - tuple). Note that the formal series ring $\mathbf{R}^{p\times m}[[s]][\hat{z}]$ is the completion of the polynomial matrix ring $\mathbf{R}^{p\times m}[s][\hat{z}]$ ($\approx\mathbf{R}^{p\times m}[s,\hat{z}]$) with respect to the I- adic topology where I is the ideal of the polynomial matrix ring $\mathbf{R}^{p\times m}[s][\hat{z}]$ generated by the indeterminate complex $(\hat{q}+\hat{q}'+1)$- tuple $(s,\hat{z})$.

Theorem 1: The following properties hold for any positive integers p, m and n:

(i) $\mathbf{R}^{n\times n}\left[e^{-h_i s}:i\in\overline{q}\cup\{0\}\right]\approx\mathbf{R}^{n\times n}\left[z^I\right]$; $\mathbf{R}^{n\times m}\left[e^{-h'_i s}:i\in\overline{q}'\cup\{0\}\right]\approx\mathbf{R}^{n\times m}\left[z^E\right]$

(ii) $\mathbf{R}^{p\times m}(s,\hat{z})\approx\mathbf{R}^{p\times m}[[s]][\hat{z}]\approx\mathbf{R}^{p\times m}[s,\hat{z}]$

(iii) $\mathbf{R}^{p\times m}[s][\hat{z}]$ is a dense subspace of $\mathbf{R}^{p\times m}[[s]][\hat{z}]$, which is a complete topological ring, with respect to the ***I***-adic topology where ***I*** is the ideal of the ring $\mathbf{R}^{p\times m}[s][\hat{z}]$ generated by the indeterminate complex $(\hat{q}+\hat{q}'+1)$- tuple $(s,\hat{z})$.

From Theorem 1 (i), the following bijections may be established:

$$\sum_{i=0}^{q}A_i e^{-h_i s}\leftrightarrow A\left(z^I\right):=\sum_{i=0}^{q}A_i z_i^I\in\mathbf{R}^{n\times n}\left[z^I\right] \tag{7}$$

$$\sum_{i=0}^{q'}B_i e^{-h'_i s}\leftrightarrow A\left(z^E\right):=\sum_{i=0}^{q}A_i z_i^E\in\mathbf{R}^{n\times n}\left[z^I\right] \tag{8}$$

So that the controllability and observability matrices of the n-th realization (1)-(2) result to be:

$$C_n(A(z),B(z))=C_n\left(A\left(z^I\right),B\left(z^E\right)\right)$$
$$:=\left(B\left(z^E\right),A\left(z^I\right)B\left(z^E\right),\ldots,A^{n-1}\left(z^I\right)B\left(z^E\right)\right)$$
$$O_n(C,A(z))=O_n\left(C,A\left(z^I\right)\right)$$
$$:=\left(C^T,A^T\left(z^I\right)C^T,\ldots,A^{n-1\,T}\left(z^I\right)C^T\right)^T \tag{9}$$

In $\mathbf{R}^{n\times(n+m)}[\hat{z}]$ and $\mathbf{R}^{p\times(p+m)}[\hat{z}]$, respectively. Define the following controllability and observability testing sets $S_{C_n}(\mathbf{h})$ and $S_{O_n}(\mathbf{h})$, respectively, depending of the real $(q+q')$ -tuple of delays $\mathbf{h}=\left(h_1,h_2,\ldots,h_q,h_{q+1}=h'_1,\ h_{q+2}=h'_2,\ldots,h_{q+q'}=h'_{q'}\right)\in\mathbf{R}_+^{q+q'}$ (the closed first orthant in $\mathbf{R}^{q+q'}$) and the associated sets of delays H_{uc} and H_{uo} where controllability and, respectively, observability are lost:

$$S_{C_n}(\mathbf{h}):=\Big\{z=\left(z_1,z_2,\ldots,z_q\right)\in\mathbf{C}^{q+q'}:z_i=|z_i|_{\prec\theta_i}=e^{-h_i s},$$
$$\sigma_C=\frac{\ln|z_i|}{h_i}\in\mathbf{R},\omega_C=\frac{tg(\theta_i)}{h_i}\in\mathbf{R},\forall i\in\overline{q+q'},$$
$$rank\left[C_n\left(A\left(z^I\right),B\left(z^E\right)\right)\right]<n\Big\}$$
$$S_{O_n}(\mathbf{h}):=\Big\{z=\left(z_1,z_2,\ldots,z_q\right)\in\mathbf{C}^{q}:z_i=|z_i|_{\prec\theta_i}=e^{-h_i s},$$
$$\sigma_O=\frac{\ln|z_i|}{h_i}\in\mathbf{R},\omega_O=\frac{tg(\theta_i)}{h_i}\in\mathbf{R},\forall i\in\overline{q},$$

$$rank\left[O_n\left(C, A\left(z^I \right) \right) \right] < n \}$$
$$H_{uc} := \left\{ \mathbf{h} \in \mathbf{R}_+^{q+q'} : S_{C_n}(\mathbf{h}) \neq \varnothing \right\}$$
$$H_{uo} := \left\{ \mathbf{h} \in \mathbf{R}_+^{q+q'} : S_{O_n}(\mathbf{h}) \neq \varnothing \right\} \tag{10}$$

If the full rank property in the above polynomial matrices are lost for some $\mathbf{h}$ such that the respective testing set is empty then controllability (respectively, observability) in a ring hold for the corresponding set of delays. If the sets $S_{C_n}(\mathbf{h})$, respectively, $S_{O_n}(\mathbf{h})$ are empty for any $\mathbf{h} \in \mathbf{R}_+^{q+q'}$ then the system is controllable (respectively, observable) in a ring independent of the delays. Note directly that:

$$\mathbf{h} \in H_{uc} \Leftrightarrow S_{C_n}(\mathbf{h}) \neq \varnothing$$
$$S_{C_n}(\mathbf{h}) = \varnothing , \forall \mathbf{h} \in \mathbf{R}_+^{q+q'} \Leftrightarrow H_{uc} = \varnothing$$

And similar assertions are applicable to the sets $S_{O_n}(\mathbf{h})$ and H_{uo}.

Theorem 2: The following properties hold:

(i) The state space realization is spectrally controllable for some given $\mathbf{h} \in \mathbf{R}_+^{q+q'}$ in the first orthant if and only if

$$\text{rank}\left[sI_n - \sum_{i=0}^{q} A_i e^{-h_i s}, \sum_{i=0}^{q'} B_i e^{-h_i' s} \right] = n, \ \forall s \in \mathbf{C} \cdot$$

(ii) The state space realization system) is spectrally observable if and only if

$$\text{rank}\left[sI_n - \sum_{i=0}^{q} A_i^T e^{-h_i' s}, C^T \right] = n, \ \forall s \in \mathbf{C} \cdot$$

(iii) The state-space realization (1)-(2) is minimal of order n if and only if it is spectrally controllable and spectrally observable; i.e.

$$rank\left[sI_n - \sum_{i=0}^{q} A_i e^{-h_i s}, \sum_{i=0}^{q'} B_i e^{-h_i' s} \right]$$
$$= \left[sI_n - \sum_{i=0}^{q} A_i^T e^{-h_i' s}, C^T \right] = n, \ \forall s \in \boldsymbol{C} \tag{11}$$

Both full rank conditions hold simultaneously then the state-space realization (1)-(2) is minimal and the converse is also true.

(iv) The state space realization is controllable in a ring independent of the delays (i.e. for any $\mathbf{h} \in \mathbf{R}_+^{q+q'}$) if $rank\left[C_n\left(A(z), B(z) \right) \right] = n, \ \forall z \in C^{q+q'}$ while the converse is not true in general. The state space realization (1)-(2) is observable in a ring independent of the delays if $rank \left[O_n\left(C, A(z) \right) \right] = n, \ \forall z \in C^{q+q'}$ and the converse is not true. The state-space realization is minimal of order n independent of the delays if: $\text{rank}\left[C_n\left(A(z), B(z) \right) \right] = \text{rank} \left[O_n\left(C, A(z) \right) \right] = n$, $\forall z \in C^{q+q'}$ while the converse is not true in general.

(v) The state space realization is controllable (respectively, observable) in a ring independent of the delays if and only if $S_{C_n}(\mathbf{h}) = \varnothing$ for any $\mathbf{h} \in R_+^{q+q'}$ (respectively, $S_{O_n}(\mathbf{h}) = \varnothing$ for any $\mathbf{h} \in R_+^{q+q'}$). The state space realization (1)-(2) is minimal if and only if $S_{C_n}(\mathbf{h}) = S_{O_n}(\mathbf{h}) = \varnothing$ for any $\mathbf{h} \in R_+^{q+q'}$; i.e. if and only if it is both controllable and observable in a ring independent of the delays.

(vi) The state space realization is controllable (respectively, observable) in a ring for any given $\mathbf{h} \in R_+^{q+q'}$ if and only if $S_{C_n}(\mathbf{h}) = \varnothing$ (respectively, $S_{O_n}(\mathbf{h}) = \varnothing$). The state space realization is minimal if and only if $S_{C_n}(\mathbf{h}) = S_{O_n}(\mathbf{h}) = \varnothing$; i.e. if and only if it is both controllable and observable in a ring.

(vii) The state space realization is controllable (respectively, observable) in a ring either dependent on $\mathbf{h} \in R_+^{q+q'}$ or independent of the delays if and only if it is spectrally controllable (respectively, spectrally observable) either dependent on or independent of the delays.

(viii) The state space realization of (1)-(2) is controllable (respectively, observable) in a ring independent of the delays, and equivalently spectrally controllable (respectively, spectrally observable) independent of the delays if and only if $H_{uc} = \varnothing$ (respectively, $H_{uo} = \varnothing$).

The state space realization (1)-(2) is minimal independent of the delays of order n if and only if $H_{uc} \cup H_{uo} = \varnothing$; i.e. if and only if it is both controllable and observable in a ring independent of the delays. The state space realization of (1)-(2) is spectrally uncontrollable (respectively, spectrally unobservable) for a given $\mathbf{h} \in \mathbf{R}_+^{q+q'}$ if and only if $\mathbf{h} \in H_{uc}$ (respectively, $\mathbf{h} \in H_{uo}$) and, equivalently, if and only if $S_{C_n}(\mathbf{h}) \neq \varnothing$ (respectively, $S_{O_n}(\mathbf{h}) \neq \varnothing$).

Now, consider the sequence $H^\tau(\hat{z}) := \left\{ H_i(\hat{z}) \right\}_1^\tau$ with $\tau \in \mathbf{N}$ which defines the τ - finite block complex Hankel matrix:

$$H\left(i, \tau+1-i, \hat{z} \right)) := \begin{bmatrix} H_1(\hat{z}) & \dots & H_{\tau+1-i}(\hat{z}) \\ \vdots & \vdots & \vdots \\ H_i(\hat{z}) & \dots & H_\tau(\hat{z}) \end{bmatrix}$$
$$= \begin{bmatrix} CB\left(z^E\right) & \dots & CA^{\tau-i}\left(z^I\right)B\left(z^E\right) \\ \vdots & \vdots & \vdots \\ CA^{i-1}\left(z^I\right)B\left(z^E\right) & \dots & CA^{\tau-1}\left(z^I\right)B\left(z^E\right) \end{bmatrix} \tag{12}$$

For $\tau = \infty$, the infinite Hankel block matrix is $H_G(\hat{z}) := \text{Block Matrix}\left(H_{i+j-1}(\hat{z}) \right)_{i,j \in \mathbf{N}}$. The subsequent technical result holds where the generic rank (denoted as gen rank) of the (.) – polynomial matrix (.) is its maximum rank reached on the overall set of values of its argument. Note that there is a natural surjective mapping $\mathbf{C}^{q+q'} \to \mathbf{C}^{\hat{q}+\hat{q}'}$ which maps each argument z into one corresponding $\hat{z}$ (z), it is irrelevant to replace the argument $\hat{z}$ by its pre-image z in all the subsequent notations and related discussions about controllability/ observability in the appropriate rings of polynomials, quasi-polynomials or series. Therefore, both arguments z and $\hat{z}$ are used indistinctly where appropriate according to convenience for clarity.

Lemma 1: The following properties hold independent of the delays:

(i)
$$H(i,\tau+1-i,\hat{z}) = O_i\left(C, A\left(z^I\right)\right) C_{\tau+1-i}\left(A\left(z^I\right),\ B\left(z^E\right)\right),$$
$\forall\, \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}$.

(ii) $\operatorname{rank}\left[H(i,\tau+1-i,\hat{z})\right] \le \operatorname{Min}(i,\tau+1-i,n)$, $\forall\, \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}$.

(iii) $\operatorname{rank}\left[H(i,\tau+1-i,\hat{z})\right] \le n$, for any $\tau, i \in \mathbf{N}$ with $\tau \ge n+i-1,\ i \ge n$, $\forall\, \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}$, where n is the order of the state-space realization.

(iv)
$$rank\left[H_G(\hat{z})\right] \le \underset{\tau \ge n+i-1,\ i \ge n,\ \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}}{gen\,rank}\left[H_G(\hat{z})\right] = \underset{\tau \ge n+i-1,\ i \ge n,\ \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}}{gen\,rank}\left[H(i,\tau+1-i,\hat{z})\right] \le n;$$
$\forall\, \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}$. Lemma 1 establishes that the rank of a τ-finite or infinite block Hankel matrix is always finite and it cannot exceed the order of given state-space realization.

Theorem 3: Consider two state-space realizations of the transfer matrix

$$R := \left(A_0, A_i\ (i \in \bar{q}),\ B_0, B_j\left(j \in \bar{q}'\right), C,\ D\right)$$
$$\bar{R} := \left(\bar{A}_0, \bar{A}_i\ (i \in \bar{q}),\ \bar{B}_0, \bar{B}_j\left(j \in \bar{q}'\right)\right),\ \bar{C},\ \bar{D} \equiv D\ ) \tag{13}$$

of respective orders n (minimal) and $\bar{n} > n$. Then, the following properties hold independent of the delays:

(i)
$$n \le \underset{\tau \ge \bar{n}+i-1,\ i \ge \bar{n},\ \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}}{gen\,rank}\left[\bar{H}_G(\hat{z})\right] \le \underset{\tau \ge \bar{n}+i-1,\ i \ge \bar{n},\ \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}}{gen\,rank}\left[\bar{H}(i,\tau+1-i,\hat{z})\right] \le \bar{n}$$
$$Min\left(\underset{\tau \ge n+i-1,\ i \ge n,\ \hat{z} \in S_{C_{\bar{n}}}(\boldsymbol{h})}{gen\,rank}\left[\bar{H}(i,\tau+1-i,\hat{z})\right],\ \underset{\tau \ge n+i-1,\ i \ge n,\ \hat{z} \in S_{O_{\bar{n}}}(\boldsymbol{h})}{gen\,rank}\left[\bar{H}(i,\tau+1-i,\hat{z})\right]\right) < \bar{n}$$
$$n \le \operatorname{Min}\left(\underset{\tau \ge n,\ \hat{z} \in S_{C_{\bar{n}}}(\mathbf{h})}{\text{gen rank}}\left[C_\tau\left(\bar{A}\left(z^I\right),\ \bar{B}\left(z^E\right)\right)\right],\ \underset{\tau \ge n,\ \hat{z} \in S_{0_{\bar{n}}}(\mathbf{h})}{\text{gen rank}}\left[O_\tau\left(\bar{A}\left(z^I\right),\ \bar{B}\left(z^E\right)\right)\right]\right) < \bar{n}$$
$$,\ \forall\, \mathbf{h} \in \mathbf{R}_+^{q+q'}$$

(ii)
$$\underset{\tau \ge n+i-1,\ i \ge n \in N,\ \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}}{\text{gen rank}}\left[H(i,\tau+1-i,\hat{z})\right] = \underset{\tau \ge n,\ \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}}{gen\,rank}\left\lfloor C_\tau\left(A\left(z^I\right),\ B\left(z^E\right)\right)\right\rfloor = \underset{\tau \ge n,\ \hat{z} \in \mathbf{C}^{\hat{q}+\hat{q}'}}{gen\,rank}\left\lfloor O_i\left(C,\ A\left(z^I\right)\right)\right\rfloor = n \tag{14}$$

(iii) None of the conditions below can hold for a complex function $z : \mathbf{C} \times \mathbf{R}_+^{q+q} \to \mathbf{C}^{q+q'}$ defined by

$$z(s,\boldsymbol{h}) = \left(z^I(s,\boldsymbol{h}),\ z^E(s,\boldsymbol{h})\right) = \left(e^{-h_1 s}, \dots, e^{-h_q s}, e^{-h_1' s}, \dots, e^{-h_q' s}\right) \tag{15}$$

associated with internal and external delays $h_i\left(i \in \bar{q}\right)$, $h_{q+j} = h_j'\left(j \in \bar{q}'\right)$.

$$\operatorname{rank}\left[H_G(\hat{z})\right] < n$$
$$\operatorname{rank}\left[H(i,\tau+1-i,\hat{z})\right] < n$$
$$\text{for any } \tau(\ge n+i-1),\ i(\ge n) \in \mathbf{N}$$
$$\operatorname{rank}\left[C_\tau\left(A\left(z^I\right),\ B\left(z^E\right)\right)\right] < n,\ \forall\, \tau(\ge n) \in \mathbf{N}$$
$$\operatorname{rank}\left[O_\tau\left(C,\ A\left(z^I\right)\right)\right] < n,\ \forall\, \tau(\ge n) \in \mathbf{N} \tag{16}$$

IV. Synthesis of Minimal Realizations

The problems of synthesis of a minimal realization, or a minimal partial realization, is formulated in terms of finding a state-space realization such that it matches a certain transfer matrix which is formally identical to a series Laurent expansion at ∞. Thus, given the sequence $H^\tau(\hat{z}) := \left\{H_i(\hat{z})\right\}_0^\tau$ with $\tau(\le \infty) \in \mathbf{N}$, find matrices $A_i \in \mathbf{R}^{n \times n}$ $\left(i \in \bar{q} \cup \{0\}\right)$, $B_i \in \mathbf{R}^{n \times m}$ $\left(i \in \bar{q}' \cup \{0\}\right)$, $C \in \mathbf{R}^{p \times n}$ provided they exist such that the following matching condition holds independent of the delays either for $\forall \tau \in \mathbf{N}$ (minimal synthesis problem) or for some finite $\tau \in \mathbf{N}$ (minimal partial realization problem):

$$G(s,\hat{z}) := C\left(sI - A_0 - \sum_{i=1}^{q} A_i z_i^I\right)^{-1}\left(B_0 + \sum_{i=1}^{q'} B_i z_i^E\right) + D = \sum_{i=0}^{\tau} H_i s^{-i} + 0\left(s^{-\tau-1}\right) \tag{17}$$

where

$$G(s,\hat{z}) \in \boldsymbol{R}^{p \times m}(s,\hat{z}) \approx \boldsymbol{R}^{p \times m}\left[\left[s\right]\right]\left[\hat{z}\right] \ni \sum_{i=0}^{\tau \le \infty} H_i(\hat{z}) s^{-i} \tag{18}$$

Such that n is as small as possible. If the minimal (respectively, partial minimal) realization synthesis problem is solvable (i.e. it has a solution) then by making the changes $z_i = e^{-h_i s}$, $z_{q+j} = e^{-h_j' s}$ $\left(i \in \bar{q},\ j \in \bar{q}'\right)$, a state-space realization is obtained so that the above description holds for $\tau \in \mathbf{N}$ (respectively, for some natural number $\tau < \infty$). If the problem is solvable then there are infinitely many minimal (respectively, partial minimal) realizations satisfying it since any nonsingular state transformation preserves the transfer matrix. In the following, the result that the McMillan degree (denoted by μ) of a rational transfer matrix coincides with that the rank of the infinite associated block Hankel matrix for $\hat{z} \in \mathbf{C}^{q+q'}$ (which is also called the McMillan degree of this one) is extended from the delay-free case.

Theorem 4: The following properties hold:

(i) The McMillan degree $n = \mu(G(s,\mathbf{h}))$ of the transfer matrix $G(s,\mathbf{h})$ is the unique order of any minimal realization of $G(s,\mathbf{h})$ and satisfies the constraints below for any set of delays being components of some given $\mathbf{h} \in \mathbf{R}_+^{q+q'}$:

$$\infty > n(\mathbf{h}) = \mu(G(s,\mathbf{h}))$$
$$= \underset{\tau \in \mathbf{N}}{\mathrm{Max}} \left(\underset{z \in S_{C_\tau}(\mathbf{h}) \cup S_{O_\tau}(\mathbf{h})}{\mu} \left(H^\tau(z) \right) \right)$$
$$= \underset{\tau \in \mathbf{N}}{\mathrm{Max}} \left(\sum_{i+j=\tau+1} \mathrm{rank}\, H(i,j,z) - \sum_{i+j=\tau} \mathrm{rank}\, H(i,j,z) : z \in S_{C_\tau}(\mathbf{h}) \cup S_{0_\tau}(\mathbf{h}) \right)$$
$$= \underset{z \in S_{C_\infty}(\mathbf{h}) \cup S_{0_\infty}(\mathbf{h})}{\mathrm{Max}} \left(\mathrm{rank}\, H(i,j,z) : n \le i \in \mathbf{N}, i+n-1 \le \tau \in \mathbf{N} \right) \tag{19}$$

(ii) The state space dimension $n_\tau(\mathbf{h})(\tau \in \mathbf{N})$ of any minimal partial realization satisfies

$$\infty > n_\tau(\mathbf{h})$$
$$= \underset{\tau \in \mathbf{N}}{\mathrm{Max}} \left(\underset{z \in S_{C_\tau}(\mathbf{h}) \cup S_{O_\tau}(\mathbf{h})}{\mu} \left(H(i, \tau+1-i, z) \right) : i \in \overline{\tau} \right) \tag{20}$$

$n_\tau(\mathbf{h}) = n(\mathbf{h})$ and then the minimal partial realization is a minimal realization for all $\tau(\ge \tau_0) \in \mathbf{N}$ and sufficiently large finite $\tau_0 \in N$ with

$$n_\tau(\mathbf{h}) = n_{\tau_0}(\mathbf{h}) = n(\mathbf{h})$$
$$= \underset{z \in S_{C_\infty}(\boldsymbol{h}) \cup S_{0_\infty}(\boldsymbol{h})}{Min} \left(rank\, H(i, \tau+1-i, z) : \tau_0 \le i \in N, i+\tau_0-1 \le \tau \in N \right) \tag{21}$$

(iii) Redefine by simplicity the delays according to $h_{q+i} = h_i' \; (i \in \overline{q}')$. Define $\mathbf{h}^{00} = 0$ and let $\mathbf{h}^{i0}$ be defined with $h_i \ne 0$ and $h_j = 0 \, (j \ne i)$ for $(i \in q+q')$. Assume that $n(\mathbf{h}^{i0}) = n_{i0} = n_0$ (some constant n_0 in **N**), $\forall i \in \overline{q}$, where

$$n_{i0} := \underset{\tau_{i0} \in \mathbf{N}}{\mathrm{Min}} \left(\sum_{i+j=\tau_{i0}+1} \mathrm{rank}\, H(i,j,\alpha_i) - \sum_{i+j=\tau_{i0}} \mathrm{rank}\, H(i,j,\alpha_i) \right)$$
$$= \underset{\tau_0 \in \mathbf{N}}{\mathrm{Min}} \left(\sum_{i+j=\tau_0+1} \mathrm{rank}\, H(i,j,\alpha_i) - \sum_{i+j=\tau_0} \mathrm{rank}\, H(i,j,\alpha_i) \right) \tag{22}$$

with $\alpha_i \in \mathbf{C}^{q+q'}$ having the i-th component distinct from unity and the remaining ones being unity, $\mathbf{h}^{i0}$ is an associate $(q+q')$- tuple of delays in $\mathbf{R}_+^{q+q'}$ with only the i-th component being nonzero and $A_i = 0$, with the remaining ones being zero and $\tau_0 := \mathrm{Max}\left(\tau_{i0} : i \in \overline{q+q'}\right)$. Then, the order for any minimal realization independent of the delays is:

$$n = n(\boldsymbol{h}) = n(\boldsymbol{h}^{i0}) = n_0 = \tau_0$$
$$= \underset{\tau(\ge \tau_0) \in \boldsymbol{N}}{Max} \left(\underset{z \in S_{C_\tau}(\boldsymbol{h}) \cup S_{O_\tau}(\boldsymbol{h})}{gen\ rank} H(i+\ell-1, \tau+\ell-i, z) \right) \tag{23}$$

$\forall \mathbf{h} \in \mathbf{R}_+^{q+q'}$ and all the matrices defining the state-space realization are independent of the delays.

Theorem 5: Consider the transfer matrix:

$$\hat{G}(s,\delta,\rho_1,\rho_2,\lambda,\tilde{h}) = \rho_1 C \left(s I_n - \delta \sum_{i=0}^{q} A_i e^{-h_i s} e^{-\tilde{h} s} \right)^{-1} \left(\sum_{i=0}^{q'} \rho_2 B_i e^{-h_i s} e^{-\tilde{h}' s} \right) + \lambda D \tag{24}$$

Parameterized in the sextuple of real scalars $\mathbf{p} := (\delta, \rho_1, \rho_2, \lambda, \tilde{h}, \tilde{h}')$, which models a perturbation of a nominal transfer matrix:

$$\hat{G}(s,\delta,\rho_1,\rho_2,\lambda,\tilde{h}) = C \left(s I_n - \sum_{i=0}^{q} A_i e^{-h_i s} \right)^{-1} \left(\sum_{i=0}^{q'} B_i e^{-h_i s} \right) + D \tag{25}$$

Parameterized by $\mathbf{p_0} := (1,1,1,1,0,0)$ and assume that the denominator quasi-polynomial and all the numerator quasi-polynomials possess principal term. If the realization is minimal then a minimal realization of the same order n is given by the original proposed one with the parametrical changes $C \to \rho_1 C$, $A_i \to \delta A_i \; (i \in \overline{q} \cup \{0\})$, $B_i \to \rho_2 A_i$, $D \to \lambda D$ and delay changes $h_i \to h_i + \tilde{h}$, $h_j' \to h_j' + \tilde{h}'$, $(i \in \overline{q} \cup \{0\}, j \in \overline{q}' \cup \{0\})$ for any finite delay perturbations $\tilde{h}$ and $\tilde{h}'$ and for any real λ if and only if $\rho_1 \rho_2 \delta \ne 0$.

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