More Suitable Values of Power and Root Mean Square (RMS) of Periodic Rectangular Wave Signal

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***Abstract*—Using Parseval's theorem, the power and root mean square (RMS) of periodic rectangular wave signal with unit amplitude, are discussed. Both values are 1W and 1v respectively. But when an attempt is made to establish their relation with area under the signal, these values somehow disagree. To address this issue, the rectangular wave signal is approached by defining a control, considering it as time dependent wave signal, then the associated power and root mean square are 1.333W and 1.1547v respectively. These new values welcome their relation with area.**

***Index Terms*—power, root mean square; parametric control; wave signal**

I. INTRODUCTION

The signals can be used to represent a physical phenomena [1], specially they are related to physical quantities like power and energy [2] captured by a physical system. In this connection Marc-Antoine Parseval 1799, presented a theorem about series, which was later applied to the Fourier series [3]. It states that the power of a signal calculated in time domain equals the power calculated in the frequency domain [4].

Since periodic rectangular linear wave signal has its own importance in signal processing. In this paper, the periodic signals with unit amplitude, are selected, and their area, power and root mean square are calculated to establish a relation. If area of a signal increases, the corresponding power and root mean square also increase by some pattern, but rectangular wave signal somehow disagree this pattern. To overcome this deficiency, an $\epsilon-$parametric wave signal is introduced to approach the rectangular wave signal, then its power and root mean square agree the established pattern of increase.

II. POWER AND ROOT MEAN SQUARE VALUE (RMS) OF A SIGNAL

The area under a T periodic signal $g(t)$ is

$$A=\int_0^T g(t)dt \tag{1}$$

It is a possible measure of its size, as it measures the amplitude along with duration, so the signal energy E_g is defined as

$$E_g=\int_0^T |g^2(t)|dt \tag{2}$$

It is the area under the squared signal. Next mean value of (2) is its power, defined by

$$P_g=\lim_{T\to\infty}\frac{1}{T}\int_0^T |g^2(t)|dt \tag{3}$$

The effective power is the root mean square (RMS) value of (3) and is given by

$$RMS=\sqrt{P_g}=\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T |g^2(t)|dt\right)^{\frac{1}{2}} \tag{4}$$

Mostly engineers and scientists consider signals in terms of frequency domain instead of time domain. The associated mathematical concept is the Parseval's theorem with Fourier series/transformation and periodic signals [5].

III. PARSEVAL'S THEOREM AND PERIODIC SIGNALS

Fourier analysis is very common in electronics. It is a mathematical tool for representing a periodic function of period T, as a summation of simple periodic functions, i.e., sines and cosines, with frequencies that are integer multiples of the fundamental frequency, $\omega=\frac{2\pi}{T}$rad/s. A periodic signal $g(t+T)=g(t)$ has Fourier expansion as [6].

$$g(t)=a_0+\sum[a_k cosk\omega t+b_k sink\omega t] \tag{5}$$

In (5), a_0, a_k and b_k are the Fourier coefficients and are given by

Manuscript received November 1, 2013; revised April 21, 2014.

doi: 10.12720/ijsps.2.2.128-131

$$\begin{cases} a_0=\dfrac{2}{T}\displaystyle\int_0^T g(t)dt\,; \\ a_{k(x)}=\dfrac{2}{T}\displaystyle\int_0^T g(x,t)\cos kwt\,dt\,; \\ b_k(x)=\dfrac{2}{T}\displaystyle\int_0^T g(x,t)\sin kwt\,dt\,; \end{cases} \tag{6}$$

By Parseval's theorem the power of a signal is [7].

$$P_g=\frac{1}{T}\int_0^T |g^2(t)|dt$$
$$=a_0{}^2+\frac{1}{2}\sum_{k=1}^{\infty}\left(a_k{}^2+b_k{}^2\right) \tag{7}$$

i.e. the power of a signal calculated in time domain equals the power calculated in the frequency domain. And by (4), the root mean square value is [7]

$$RMS=\left(a_0{}^2+\frac{1}{2}\sum_{k=1}^{\infty}\left(a_k{}^2+b_k{}^2\right)\right)^{\frac{1}{2}} \tag{8}$$

Using (1), (7) and (8), the area, power and root mean square of different signals are calculated to study how power and root mean square are increasing as area is increasing [8]. These signals are divided in two categories namely parametric and non-parametric. First consider nonparametric signals.

IV. Non-Parametric Signals

In this section, some common linear periodic wave signals with unit amplitude, are considered. The idea is to establish a relation of root mean square/power with area.

A. Triangular Wave Signal

First of all, consider a periodic triangular wave signal: $R_s(t)=R_s(t+T)$ (see Fig. 1) given by (9)

$$R_s(t)=\begin{cases} \dfrac{4}{T}t & if\ 0\le t<\dfrac{T}{4} \\ \dfrac{4}{T}\left(-t+\dfrac{T}{2}\right) & if\ \dfrac{T}{4}\le t<\dfrac{3T}{4} \\ \dfrac{4}{T}t & if\ \dfrac{3T}{4}\le t<T \end{cases} \tag{9}$$

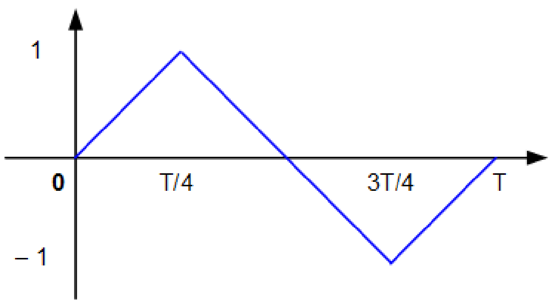

Figure 1. Triangular wave signal

Using (1), area under this signal is given by (10)

$$A_{(RS)}=0.5\,T\ units^2 \tag{10}$$

Using (6), the Fourier coefficients of (9) are

$$\begin{cases} a_0=0 \\ a_k=0 \\ b_k=\dfrac{8}{\pi^2}\left(\dfrac{(-1)^k}{(2k+1)^2}\right) \end{cases} \tag{11}$$

Using (11) in (7), the power associated with this signal is

$$P_{(RS)}=0.5\sum_{k=1}^{\infty}\left(\frac{8}{\pi^2}\left(\frac{(-1)^k}{(2k+1)^2}\right)\right)^2$$
$$=0.3334\ W \tag{12}$$

And by (8) its effective power is

$$RMS_{(RS)}=0.5774\ \text{v} \tag{13}$$

The power and root mean square with triangular wave signal are given by (12) and (13) respectively.

B. Inclined Wave Signal

Next consider a periodic inclined wave signal, shown in Fig. 2, given by (14)

$$R_I(t)=-\frac{2}{T}t+1; \qquad if\ 0\le t<T \tag{14}$$

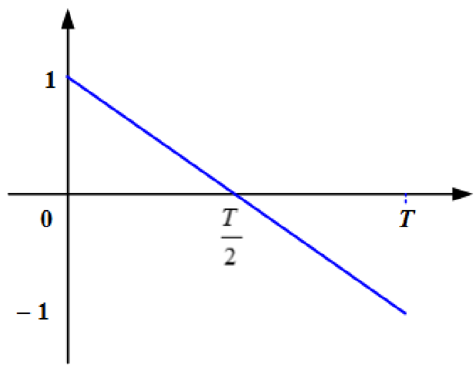

Figure 2. Inclined wave signal

The area under this signal is given by (15)

$$A_{(i)}=0.5\ T\ units^2 \tag{15}$$

And by (6), the Fourier coefficients are

$$\begin{cases} a_0=0 \\ a_k=0 \\ b_k(x)=\dfrac{2}{k\pi} \end{cases} \tag{16}$$

Using (16) in (7), the power associated with this signal is

$$P_{(i)}=0.3334\ W \tag{17}$$

And by (8) its effective power is

$$RMS_{(i)}=0.5774\ \text{v} \tag{18}$$

The power and root mean square with inclined wave signal are given by (17) and (18) respectively. This signal has same power and root mean square value as above. The signals in subsections *A* and *B* have same area, so have same power and root mean square values.

C. Quadratic Wave Signal

Next define a periodic quadratic wave signal: $Q_c(t)$ (see Fig. 3), defined by (19)

$$Q_c(t)=\begin{cases} 1 & if\ 0\leq t<0.4\ T \\ \frac{5}{T}(-2t+T) & if\ 0.4\ T\leq t<0.6\ T \\ -1 & if\ 0.6\ T\leq t<T \end{cases} \quad (19)$$

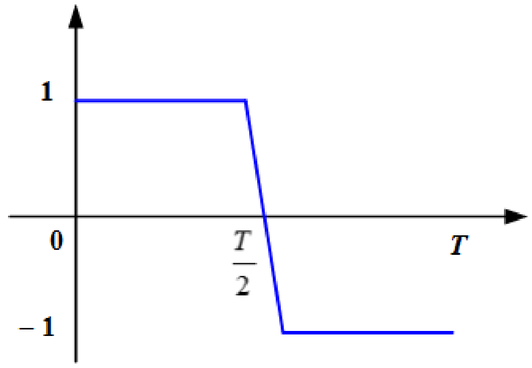

Figure 3. Quadratic wave signal

The area under this signal is given by (20)

$$A_{(Q_c)}=0.9\ T\ units^2 \quad (20)$$

The Fourier coefficients are

$$\begin{cases} a_0=0 \\ a_k=0 \\ b_k(x)=\left(\frac{2}{k\pi}+\frac{10}{k^2\pi^2}\sin(0.2k\pi)\right) \end{cases} \quad (21)$$

Using (21) in (7), the power associated with this signal is

$$P_{(RS)}=0.5\sum_{k=1}^{\infty}\left(\frac{2}{k\pi}+\frac{10}{k^2\pi^2}\sin(0.2k\pi)\right)^2$$
$$=1.0247\ W \quad (22)$$

And by (8) the corresponding effective power is

$$RMS_{(Q_c)}=1.0123\ \text{v} \quad (23)$$

The power and root mean square with quadratic wave signal are given by (22) and (23) respectively.

As area increases, the power and root mean square also increase.

D. Rectangular Wave Signal

Next introduce a periodic rectangular wave signal: $R_l(t)$ (see Fig. 4), defined by (24)

$$R_l(t)=\begin{cases} 1 & if\ 0\leq t<0.5\ T \\ -1 & if\ 0.5\ T\leq t<\ T \end{cases} \quad (24)$$

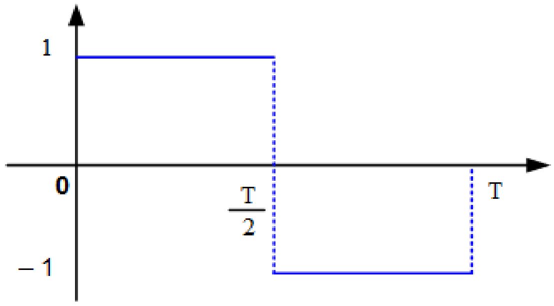

Figure 4. Rectangular wave signal

The area under this signal is given by (25)

$$A_{(l)}=T\ units^2 \quad (25)$$

The Fourier coefficients are

$$\begin{cases} a_0=0 \\ a_k=0 \\ b_k(x)=\frac{4}{\pi(2k-1)} \end{cases} \quad (26)$$

Using (26) in (7), the power associated with this signal is

$$P_{(l)}=0.5\sum_{k=1}^{\infty}\left(\frac{4}{\pi(2k-1)}\right)^2$$
$$=1W \quad (27)$$

and by (8) the corresponding effective power is

$$RMS_{(l)}=1\ \text{v} \quad (28)$$

The power and root mean square with quadratic wave signal are given by (27) and (28) respectively.

Very interesting, the rectangular wave signal has more area than quadratic wave signal, but its power and root mean square are less than it. See the quadratic wave signal has 1.8 times more area than triangular/inclined wave signal; also its root mean square is almost 1.8 times more than them. Next the rectangular wave signal has 2 times more area than triangular/inclined wave signal, but its root mean square is not 2 times more than them. It means the power and root mean square of rectangular wave signal are not accordingly. For this an $\epsilon-$parametric control is defined, considering signal similar to (19). Its asymptotic case ($\epsilon\rightarrow 0$) is approaching to rectangular wave signal.

V. Parametric Rectangular Signal

This $\epsilon-$parametric control with $0<\epsilon<1$ is given by (29) and illustrated in the Fig. 5.

$$R_\epsilon(t)=\begin{cases} 1 & if\ 0\leq t<\frac{1-\epsilon}{2}T \\ \frac{1}{\epsilon}\left(-\frac{2}{T}t+1\right) & if\ \frac{1-\epsilon}{2}T\leq t<\frac{1+\epsilon}{2}T \\ -1 & if\ \frac{1+\epsilon}{2}T\leq t<T \end{cases} \quad (29)$$

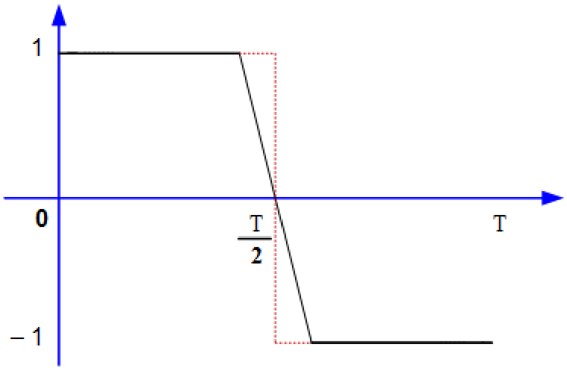

Figure 5. ϵ −Parametric quadratic wave signal

The area under this signal is given by (30)

$$A_{(1)}=(1- 0.5\epsilon)T\ units^2 \tag{30}$$

The Fourier coefficients are

$$\begin{cases} a_0=0 \\ a_k=0 \\ b_k(x)=\left(\dfrac{2}{k\pi}+\dfrac{2}{\epsilon k^2\pi^2}\sin(\epsilon k\pi)\right) \end{cases} \tag{31}$$

Using (31) in (7), the power associated with this signal is

$$P_{(1)}=0.5\sum_{k=1}^{\infty}\left(\frac{2}{k\pi}+\frac{2}{\epsilon k^2\pi^2}\sin(\epsilon k\pi)\right)^2 \tag{32}$$

From (32) the power of a signal can be calculated, selecting ϵ from $0<\epsilon 1$. For $\epsilon\rightarrow 0$ this signal approaches to rectangular signal. In this case its area is given by (33)

$$A_{(\epsilon\rightarrow 0)}=T\ units^2 \tag{33}$$

This area is same as that of rectangular wave signal. The associated power is

$$P_{(\epsilon\rightarrow 0)}=\lim_{\epsilon\rightarrow 0}\left(0.5\sum_{k=1}^{\infty}\left(\frac{2}{k\pi}+\frac{2}{\epsilon k^2\pi^2}\sin(\epsilon k\pi)\right)^2\right)$$
$$=1.3333\ W \tag{34}$$

And the corresponding root mean square value is

$$RMS_{(\epsilon\rightarrow 0)}=1.1547\ \text{v} \tag{35}$$

The power and root mean square with quadratic wave signal are given by (34) and (35) respectively.

The power and root mean square value are more than the existing values of rectangular wave signal. Also its new root mean square value is 2 times more than the value of triangular/inclined wave signal. Hence its new values of power and root mean square can be considered accordingly.

VI. Conclusion

Using Parseval's theorem, an attempt is made to agree that the power and root mean square (*RMS*) of rectangular wave signal with unit amplitude are $1.3333\ W$ and $1.1547\ v$ respectively. In this regard first argument is made to establish the direct relation of power and root mean square value with area by introducing different waveform signals. All signals agree a pattern of increase except rectangular wave signal. In next argument, rectangular wave signal is approached by defining a control, and then it follows the established pattern of increase.

Moreover these control strategies provide a direction to consider time dependent signals instead of time independent signals.

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